THE

Young Mathematician's Guide.

Being a PLAIN and EASIE

INTRODUCTION

TO THE

Mathematicks.

In FIVE PARTS.

VIZ.

I. Arithmetick, Vulgar, and Decimal, with all the Useful Rules; And a General Method of Extracting the Roots of all Single Powers.

II. Algebra, or Arithmetick in Species; wherein the Method of Raising and Resolving Æquations is rendered Easie; and Illustrated with Variety of Examples, and Numerical Questions. Also the whole Business of Interest and Annuities &c. perform'd by the Pen.

III. The Elements of Geometry, Contracted, and Analytically Demonstrated; With a New and Easie Method of finding the Circle's Periphery and Area to any assigned Exactness, by one Æquation only; Also a New way of making Sines and Tangents.

IV. Conic-Sections, wherein the Chief Properties, &c. of the Ellipsis, Parabola, and Hyperbola, are clearly demonstrated.

V. The Arithmetick of Infinites explain'd, and render'd Easie; with its Application to Superficial, and Solid Geometry.

With an APPENDIX of Practical Gaging.

The Sixth Edition, *carefully Corrected; and New Tables of Compound Interest at Five* per Cent. *Calculated, and Added by the Author,*

JOHN WARD.

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To the Honourable

Sir *RICHARD GROSVENOR* of *Eaton*, in the County *Palatine* of *Chester*, Baronet.

SIR,

WHEN requeſted by ſome Bookſellers in *London*, to Reviſe and Prepare this Treatiſe for a New Impreſſion, and once reſolved to Anſwer their Demands; I was not long conſidering at whoſe Feet to lay it.

My Memory may indeed be impair'd by Age, Miſfortunes and Accidents; Nay, I am ſenſible it is ſo: But it muſt be entirely loſt, when I am forgetful of the great Obligations, I lie under to Sir *Richard Groſvenor*.

Your Hoſpitality and Generoſity, make you ſtand unenvied in the Abundance of Fortune. Any Upſtart may contrive to ſpend a great Eſtate: But it is a Felicity almoſt peculiar to great Birth to become One.

Were I now to deſcribe Liberality without Profuſeneſs; Steadineſs in Principles, without any private View; Candor and Affability, good Nature join'd to ſound Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; And then pronounce that you are poſſeſſed of all theſe good Qualities in as high a Degree as moſt Men living; No Gentleman that knows you well, would think I flatter'd you.

 Sir,

Sir, Give me Leave to say, I Honour your Character, and Love your Person; My Expressions are uncourtly, my Stile unpolish'd, and therefore more proper to be prefix'd to a Work, wherein the Matters related are indeed clad in a plain and homely Dress; but they are True, and designed to propagate Mathematical Learning amongst such as desire to be introduced into that sort of Knowledge; And I am extremely pleas'd they are permitted to be sent into the World under your Protection.

That you may long Live, to promote the Good of your Country, and that City in whose Interest you have so heartily engag'd your Self; And that you may ever succeed in your own private Affairs; And live to enjoy all the Blessings that attend a quiet prudent Life, Is the earnest Prayer of,

Honoured SIR,

Your most Obliged, Humble,

and Obedient Servant,

Thomas
his Book August
the 6th 1815 —

J. WARD.

To

To the READER.

I *Think it needleſs* (and almoſt endleſs) *to run over all the Uſefulneſs, and Advantages of* Mathematicks *in General; ſhall therefore only touch upon thoſe Two Admirable Sciences* Arithmetick, *and* Geometry; *which are indeed the Two grand Pillars* (or rather the Foundations) *upon which all other parts of Mathematical Learning depend.*

As to the Uſefulneſs of Arithmetick, *'Tis well known that no Buſineſs, Commerce, Trade, or Imployment whatſoever, even from the Merchant to the Shop-keeper, &c. can be manag'd and carry'd on, without the Aſſiſtance of Numbers.*

And as to the Uſefulneſs of Geometry, *'Tis as certain, that no Curious Art, or Mechanick-Work, can either be invented, improved or performed, without its aſſiſting Principles, tho' perhaps the Artiſt, or Workman, has but little* (nay ſcarce any) *Knowledge in* Geometry.

Then, as to the Advantages that ariſe from both theſe Noble Sciences, when duly join'd together, to aſſiſt each other, and then apply'd to Practice (according as occaſion requires) *will readily be granted by all who conſider the vaſt Advantages that accrue to Mankind from the Buſineſs of Navigation only. As alſo from that of Surveying and dividing of Lands betwixt Party and Party. Beſides the great Pleaſure and Uſe there is from Time-keepers, as Dials, Clocks, and Watches, &c. All theſe, and a great many more very uſeful Arts,* (too many to be enumerated here) *wholly depend upon the aforeſaid Sciences.*

And therefore 'tis no Wonder, that in all Ages ſo many Ingenious, and learned Perſons have imploy'd themſelves in writing upon the Subject of Mathematicks; *but then moſt of thoſe Authors ſeem to preſuppoſe that their Readers had made ſome Progreſs in that ſort of Learning before they attempted to peruſe thoſe Books, which are generally large Volumns, written in ſuch abſtruſe Terms that young Learners were really affraid of looking into thoſe Studies.*

Theſe

The PREFACE.

These Considerations first put me (many Years ago) *upon the Thoughts of endeavouring to compose such a plain, and familiar Introduction to the* Mathematicks, *as might Encourage those that were willing* (to spend some Time that way) *to venture and proceed on with Chearfulness; Tho' perhaps they were wholly ignorant of its first Rudiments. Therefore I began with their first Elements or Principles.*

That is, I began with an Unit in Arithmetick, and a Point in Geometry; And from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I can, &c.

And for that Reason I published this Treatise (Anno 1707) *by the Title of the* Young Mathematician's Guide; *which has answer'd the Title so well, that I believe I may truly say* (without Vanity) *this Treatise hath prov'd a very helpful Guide to near Five Thousand Persons; and perhaps most of them such as would never have look'd into the* Mathematicks *at all, but for it.*

And not only so, but it hath been very well receiv'd amongst the Learned, and (I've been often told) *so well Approv'd on at the Universities, in* England, Scotland, *and* Ireland, *that it's Order'd to be publickly read to their Pupils,* &c.

The Title Page gives a short Account of the several Parts treated of, with the Corrections and Additions that are made to this Sixth Edition, *which I shall not inlarge upon, but leave the* Book *to speak for it self; and if it be not able to give Satisfaction to the Reader, I'm sure, all I can here say in its behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: 'Tis true indeed, the Dress is but Plain and Homely, it being wholly intended to Instruct, and not to Amuse or Puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I've sincerely aim'd at what's useful, tho' in one of the meanest Ways; 'Tis Honour enough for me to be accounted as one of the Under-Labourers in Clearing the Ground a little, and removing some of the Rubbish that lay in the Way to this Sort of Knowledge. How well I have perform'd That, must be left to proper Judges.*

To be brief; As I am not sensible of any Fundamental Error in this Treatise; so I will not pretend to say, it is not without Imperfections, (Humanum est errare) *which I hope the Reader will excuse, and pass them over with the like Candor and Good Will that it was composed for his Use; by his real Well-wisher,*

J. WARD.

THE

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Chap.

AN
INTRODUCTION
TO THE
Mathematicks.

PART I.

PRÆCOGNITA.

THE Business of *Mathematicks* in all its Parts both *Theory* and *Practice*, is only to search out and determine the true Quantity; either of *Matter*, *Space*, or *Motion*, according as Occasion requires.

By *Quantity* of *Matter* is here meant the *Magnitude* or Bigness of any visible Thing, whose *Length*, *Breadth*, and *Thickness* may either be measured, or estimated.

By *Quantity* of *Space* is meant the Distance of one Thing from another.

And by *Quantity* of *Motion* is meant the Swiftness of any Thing moving from one Place to another.

The Consideration of these according as they may be proposed, are the *Subjects* of the *Mathematicks*, but chiefly that of *Matter*.

Now the Consideration of *Matter*, with respect to its *Quantity*, *Form* and *Position*, which may either be *Natural*, *Accidental*, or *Designed*, will admit of infinite Varieties; but all the Varieties that are yet known, or indeed possible to be conceived, are wholly comprised under the due Consideration of these two, *Magnitude* and *Number*, which are the proper Subjects of *Geometry*, *Arithmetick* and *Algebra*. All other *Parts* of the *Mathematicks* being only the Branches of these three *Sciences*, or rather their Application or particular Cases.

GEOMETRY is a *Science* by which we ſearch out and come to know either the whole *Magnitude* or ſome Part of any propoſed *Quantity*; and is to be obtained by comparing it with another known *Quantity* of the ſame kind, which will always be one of theſe, *viz.* a *Line* (or *Length* only) a *Surface*, (that is, *Length* and *Breadth*) or a *Solid* (which hath *Length*, *Breadth* and *Depth*, or *Thickneſs*) Nature admitting of no other Dimenſions but theſe three.

ARITHMETICK is a *Science* by which we come to know what *Number* of *Quantities* there are (either real or imaginary) of any Kind,contained in another *Quantity* of the ſame Kind: Now this Conſideration is very different from that of *Geometry*, which is only to find out true and proper Anſwers to all ſuch Queſtions as demand, how *Long*, how *Broad*, how *Big*, &c. But when we are to conſider either of more *Quantities* than *one*, or how often one *Quantity* is contained in another, then we have recourſe to *Arithmetick*, which is to find out true and proper Anſwers to all ſuch Queſtions as demand how *Many*, what *Number*, or *Multitude* of *Quantities* there are. To be brief the Subject of *Geometry* is that of *Quantity*, with reſpect to its *Magnitude* only; and the Subject of *Arithmetick* is *Quantities* with reſpect to their *Number* only.

ALGEBRA is a *Science* by which the moſt abſtruſe or difficult Problems either in *Arithmetick* or *Geometry* are reſolved and demonſtrated, that is, it equally interferes with them both; and therefore it is promiſcuouſly named, being ſometimes called *Specious Arithmetick*, as by *Harriot*, *Vieta*, and Doctor *Wallis*, &c. And ſometimes it is called *Modern Geometry*, particularly the ingenious and great Mathematician, Mr. *Edmond Halley*, Savilian Profeſſor of *Geometry* in the Univerſity of *Oxford*, giving this following Inſtance of the Excellence of our *Modern Algebra*, writes thus,

'The Excellence of the *Modern Geometry* (ſaith he) is in nothing 'more evident, than in thoſe full and *Adequate Solutions* it gives to '*Problems*: Repreſenting all the poſſible Caſes at one View, and 'in one general *Theorem* many times comprehending whole *Sciences*; 'which deduced at length into *Propoſitions* and demonſtrated after 'the Manner of the *Ancients*, might well become the Subjects of 'large Treatiſes: For whatſoever *Theorem* ſolves the moſt compli'cated *Problem* of the Kind, does with a due Reduction reach all 'the ſubordinate *Caſes*. Of which he gives a notable Inſtance in the *Doctrine* of *Dioptricks* for finding the *Foci* of *Optick Glaſſes* univerſally, (vid. *Philoſophical Tranſactions*, Numb. 205.)

Thus you have a ſhort and general Account of the proper Subjects of thoſe three noble and uſeful *Sciences*, *Arithmetick*, *Geometry* and *Algebra*. I ſhall now proceed to give a particular Account of each, and firſt of *Arithmetick*, which is the *Baſis* or *Foun-*

dation

dation of all *Arts*, both *Mathematick* and *Mechanick*; and therefore it ought to be well underſtood before the reſt are medled withal.

CHAP. I.

Concerning the ſeveral Parts of Arithmetick, *with the Definition of ſuch Characters as are uſed in this Treatiſe.*

ARITHMETICK, or the Art of Numbering, is fitly divided into three diſtinct Parts, two of which are properly called Natural, and the third Artificial.

The Firſt being the moſt plain and eaſieſt, is commonly called Vulgar Arithmetick in whole Numbers; becauſe every Unit or Integer concerned in it, repreſents one whole Quantity of ſome Species or Thing propoſed.

The Second is that which ſuppoſes an Unit (and conſequently the Quantity or Thing repreſented by that Unit) to be broken or divided into equal Parts (either even, or uneven) and conſiders of them either as pure Parts, *viz.* each leſs than an Unit, or elſe of Parts and Integers intermixt. And is uſually called the Doctrine of Vulgar Fractions.

The Third, or Artificial Part, is called Decimal Arithmetick; being an Artificial Invention of managing Fractions or broken Numbers, by a much more commodious and eaſy Way than that of Vulgar Fractions: For the ſeveral Operations performed in Decimals, differ but little from thoſe in whole Numbers; and therefore it is now become of general Uſe, eſpecially in Geometrical Computations.

ARITHMETICK (in all its Parts) is performed by the various ordering and diſpoſing of ten *Arabick* Characters or Numeral Figures (which by ſome are called Digits)

viz.	*One*	*Two*	*Three*	*Four*	*Five*	*Six*	*Seven*	*Eight*	*Nine*	*Cypher.*
	1	2	3	4	5	6	7	8	9	0

The Uſe of the Characters is ſaid to be firſt introduced into *England* near ſix hundred Years ago, *viz.* about the Year 1130. vide Doctor *Wallis*'s Algebra, *Page* 12.

The firſt of theſe Characters is called Unity, and repreſents one of any Kind of Species or Quantity. As one World, one Star, one Man, *&c.*

Viz. Unity is that by which every Thing that is, is called one (*Euclid.* 7. *Def.* 1.) and is the Beginning of all Numbers. That is to ſay, Number is a Multitude of Units. *Euclid.* 7. *Def.* 2.

For, one more one, makes two; and one, more one, more one makes three, *&c.* Which is the firſt and chief *Poſtulate,* or rather *Axiom* to Arithmetick.

Viz. { That $1+1=2$. $1+1+1=3$. $1+1+1+1=4$. $1+1+1+1+1=5$. And ſo on to 9.

Nine of theſe Figures were thus compoſed of Units, and differently form'd to repreſent ſo many Units put together into one Sum, as was intended each ſhould denote: Nine being the greateſt Number of Units that was then thought convenient to be expreſſed by one ſingle Character; the laſt of the Ten is only a Cypher, or (as ſome phraſe it) a Nothing, becauſe of it ſelf it ſignifies nothing; for if never ſo many Cyphers be added to, or ſubtracted from any Number, they can neither increaſe nor diminiſh that Number; but yet as a Cypher (or Cyphers) may be placed, the other Figures will become of different Values from what they were before, as will appear further on.

For the more convenient Ordering of the aforeſaid Numeral Figures according to the ſeveral Varieties that happen in Computations; I do adviſe the young Learner to acquaint himſelf with the Signification of the following Algebraick Signs or Characters, which he will find of excellent Uſe, as being a much ſhorter, better and more ſignificant Way of denoting what is to be done (in moſt Operations) than can otherwiſe be expreſſed in Words at length.

Significations.

Signs	*Names.*	
$+$	*Plus* or *more.*	The Sign of *Addition*; As $8+7$ is 8 more 7, and ſignifies that the Numbers 8 and 7 are to be added into one Sum. The like is to be underſtood when ſeveral Numbers are connected together with the Sign $+$. As $34+22+9+45$, *&c.* denotes theſe are all to be added into one Sum.

The

$-$	Minus or less.	The Sign of *Subtraction*; As $9 - 6$ is 9 less 6, and signifies that 6 is to be taken from 9, that so their Difference may be found.
$\times$	Into or with.	The Sign of *Multiplication*; As 9×6, is 9 into 6, and signifies that 9 is to be multiplied into or with 6.
$\div$	By.	The Sign of *Division*; As $8 \div 2$, is 8 by 2, and signifies that 8 is to be divided by 2, also thus 2) 8 (4. or thus $\frac{8}{2}$, each signifying the same Thing, to wit, 8 divided by 2.
$=$	Equal.	The Sign of *Equality* or *Equation*, *viz.* whenever this Sign $=$ is placed betwixt Numbers (or Quantities) it denotes them to be equal, as $9=9$, or $9+6=15$, or $9-6=3$, &c. That is, 9 is equal to 9, or 9 more 6, is equal to 15, and 9 less 6, is equal to 3, &c.
$: :$	So is.	The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and : : is always placed betwixt the two middle Terms or Numbers in Proportion. Thus, 2 : 8 : : 6 : 24. To be read thus; As 2, is to 8, so is 6, to 24.

These Signs and their Significations, being perfectly learnt, will help to shorten the Work.

CHAP. II.

Concerning the principal Rules in Arithmetick, *and how they are performed in whole Numbers.*

THE Rules by which Numerical Operations are perform'd in all the Parts of Arithmetick, are many and various, several of them being form'd and raised as Occasion requires, when applied to Practice, yet they are all comprehended within the due Consideration of these Six, *viz.* NUMERATION (or NOTATION

NOTATION, ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, and EVOLUTION, or *Extraction* of *Roots*.

Sect. I. *Of* Numeration, *or* Notation.

NUMERATION or Notation, teacheth to read or express the true Value of any Number when writ down; and consequently to write down any proposed Number according to its true Value when it is named: And this consisteth of two Parts;

1. The due Order of placing down Figures.
2. The true Valuing of each Figure in its Place.

Both which are plainly exhibited in the following Table.

&c.	*Hundreds of Thousands of Millions.*	*Tens of Thousands of Millions.*	*Thousands of Millions.*	*Hundreds of Millions.*	*Tens of Millions.*	*Millions.*	*Hundreds of Thousands.*	*Tens of Thousands.*	*Thousands.*	*Hundreds.*	*Tens.*	*Units* } *And is the first Place in counting.*
	6	7	8	9	8	7	6	5	4	3	2	1
	Period of Thousands of Millions.			*Period of Millions.*			*Period of Thousands.*			*Period of Units.*		

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By this *Numeration* Table it's apparent that the Order of Places is reckoned from the Right-Hand towards the Left; the first Place of any Number being always that which is the outmost Figure to the Right-Hand; and whatever Figure stands in that Place doth only signify its own simple Value, *viz.* so many Units as that Figure represents.

The second Place is that of Tens, any Figure standing in that Place signifieth so many Tens as that Figure represents Units. The

The third Place is Hundreds, the fourth Place Thouſands, &c. That is, each Place towards the Left-Hand is ten times the Value of that next it towards the Right.

For Inſtance, ſuppoſe 759 were propoſed to be read or pronounced according to the Value of each Figure as they now ſtand. The firſt Figure in this Sum is 9, becauſe it ſtands in the Place of Units, and therefore ſignifies but its own ſimple Value, to wit 9 Units, or Nine. The ſecond Figure 5 ſtands in the Place of Tens, and therefore ſignifies five Tens or Fifty. The Figure 7 ſtands in the third Place, or Place of Hundreds, and therefore it ſignifies ſeven Hundred, and the whole Sum is to be read or pronounced thus, Seven Hundred Fifty Nine.

Note, Although the Figure 7, ſtands in the third Place (according to the Order of Numbering) yet when the whole Sum comes to be read it is firſt pronounced, the reading of Numbers being perform'd like that of Letters or Words, always beginning with the outmoſt Figure towards the Left-Hand, and ſo many Figures as are placed together without any Point, Comma, Line, or other Note of Diſtinction between them, are all but one Sum, and muſt be read as ſuch.

For Example, 763596 is but one intire Sum or Number, notwithſtanding it conſiſts of ſix Places of Figures, and is thus read; Seven Hundred Sixty Three Thouſand Five Hundred Ninety Six.

The like is to be obſerved in reading or expreſſing the true Value of any Sum or Rank of Numbers conſiſting of Seven, Eight, Nine, or more Places of Figures, each Figure being to be valued according to its Diſtance from the Place of Unity: As in the foregoing Table.

Now ſuch Values may as well ariſe by *Cyphers*, as by other Figures; for Inſtance, 6 ſtanding by it ſelf, repreſents but ſix Units: But if a Cypher be annext to it thus, 60, then it becomes Sixty; for the Cypher poſſeſſing the Place of Units, hath thereby removed the 6 into the Place of Tens; and another Cypher more would make it 600, Six Hundred, &c.

Whence it may be noted, that altho' a Cypher of it ſelf ſignify nothing (as hath been ſaid before) yet being placed on the Right-Hand of any Figure, it augments the Value of that Figure by advancing it into a higher Place than otherwiſe it would have been, had not the Cypher been there.

Take one Example more in Numeration, if you pleaſe, that in the Table *viz.* 678987654321, which is, according as is there ſignified,

Six Hundred Seventy Eight Thouſand Millions,
Nine Hundred Eighty Seven Millions,
Six Hundred Fifty Four Thouſand,
Three Hundred Twenty One Units. Of any propoſed Species or Quantities whatſoever.

And

And here it may be obſerved, that every third Figure from the Place of Units, bears the Name of Hundreds; which ſhews that if any great Sum be parted, or rather diſtinguiſhed into Periods, of three Figures in each Period (as in the foregoing Table) it will be of good Uſe to help the young Learner in the eaſier valuing and expreſſing that Sum.

Sect. 2. *Of* Addition.

Poſtulate or Petition.

That any given Number may be increaſed or made more, by putting another Number to it.

ADDITION is that Rule by which ſeveral Numbers are collected and put together, that ſo their Sum or total Amount may be known.

In this Rule two Things being carefully obſerved, the Work will be eaſily performed.

1. The firſt is the true placing of the Numbers, ſo as that each Figure may ſtand directly underneath thoſe Figures of the ſame Value, *viz.* Place Units under Units, Tens under Tens, and Hundreds under Hundreds, &c.

Then underneath the loweſt Rank (always) draw a Line to ſeparate the given Numbers from their Sum when it's found.

Example. If theſe Numbers 54327, and 2651, were given to be added together, they muſt be placed

$$\text{Thus,}\left\{\begin{array}{r}54327\\2651\\\hline\end{array}\right.$$

2. The ſecond Thing to be obſerved is the due collecting or adding together each Row of Figures that ſtand over one another of the ſame Value: And that is thus performed.

RULE.

Always begin your Addition at the Place of Units, and add together all the Figures that ſtand in that Place, and if their Sum be under Ten, ſet it down below the Line underneath its own Place; but if their Sum be more than Ten, you muſt ſet down only the Overplus, or odd Figure above the Ten (or Tens) and ſo many Tens as the Sum of thoſe Units amount to, you muſt carry to

to the Place of Tens; Adding them and all the Figures that ſtand in the Place of Tens together, in the ſame Manner as thoſe of the Units were added; then proceed in the ſame Order to the Place of Hundreds, and ſo on to each Place until all is done.

The Sum ariſing from thoſe Additions will be the total Amount required.

Example 1.

Let it be required to find the Sum of the aforeſaid Numbers.

viz. { 54327
　　　 2651

56978 the Sum required.

Beginning at the Place of Units, I ſay 1 and 7 is 8, which being leſs than 10, I ſet it down (according to the Rule) underneath its own Place of Units; and then proceed to the Place of Tens, ſaying 5 and 2 is 7, which being leſs than 10, I ſet it down underneath its own Place of Tens, and proceed to do the like at the Place of Hundreds, and then at Thouſands, ſetting each of their Sums underneath their own reſpective Places: Laſtly, Becauſe there is not any Figure in the lower Rank to be added to the Figure 5, which ſtands in the Place of Ten Thouſands, in the upper Rank, I therefore bring down the ſaid 5 to the reſt, placing it underneath its own Place, and then I find that $54327 + 2651 = 56978$, the true Sum required.

Example 2.

Suppoſe it were required to find the Sum of theſe Numbers, $3578 + 496 + 742 + 184 + 95$. Theſe being placed, as before directed, will ſtand as in the Margin. Then beginning (as before) at the Place of Units, ſay 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25, ſet down the 5 Units underneath its own Place of Units, and carry the 20, or two Tens, to the Place of Tens (at which Place they are only 2) ſaying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39, ſet down the 9 underneath its own Place of Tens, and carry the 30, or three Tens (which indeed is 300) to the Place of Hundreds, at which Place they are but 3, ſaying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20, here becauſe there is no Figure overplus (as before) I ſet down a Cypher underneath the Place of Hundreds, and carry the 2 Tens (or rather the 2000) to the Place of Thouſands, ſaying

3578
496
742
184
95
————
5095

C　　　　(as

(as before,) 2 I carry and 3 is 5, which being the laſt, I ſet it down underneath its own Place, and all is finiſhed. And find the Sum or total Amount to be $5095 = 3578 + 496 + 742 + 184 + 95$.

If this Example be well conſidered, it will be ſufficient to ſhew the uſual Method of *Addition* in Whole *Numbers*; but to make all plain and clear, I ſhall ſhew the young Learner the Reaſon of carrying the Tens from one Degree or Row of Figures, to the next Superior Degree, which is done purely to ſave Trouble, and prevent the uſing of more Figures than are really neceſſary, as will appear by the following Method of adding together the ſame Numbers of the laſt Example.

Thus, Add together each ſingle Row of Figures by it ſelf; as if there were no more but that one Row, ſetting down the Sum underneath its own Place.	3578 496 742 184 95
The Sum of the Row of Units, is	25
The Sum of the Row of Tens, is	370
The Sum of the Row of Hund. is	1700
The three Thouſand brought down	3000
The Sum or total Amount as before, is	5095

Add

From hence I preſume it will be eaſy to conceive the true Reaſon of carrying the aforeſaid Tens; and alſo that Cyphers do not augment or increaſe the Sum in Addition. (See *Page* 4.)

I might have here inſerted a Lineal Demonſtration of this Rule of Addition; but I thought it would rather puzzle than improve a young Learner, eſpecially in this Place; beſides the Reaſon of it is ſufficiently evident from that Natural Truth of *the Whole being equal to all its Parts taken together.* Euclid. 1. Axiom 19.

That is, the Numbers which are propoſed to be added together are by that *Axiom* underſtood to be the ſeveral Parts, and their Sum or total Amount found by Addition is underſtood to be the Whole,

And from thence is deduced the Method of proving the Truth of any Operation in Addition, *viz.* By parting or ſeparating the given Numbers into two Parcels (or more, according to the largeneſs of it) and then Adding up each Parcel by it ſelf: For if thoſe particular Sums ſo found, be Added into one Sum, and that Sum prove equal or the ſame with the total Sum firſt found;

found, then all is right; if not, care muſt be taken to diſcover and correct the Error.

Example.

Add	5647 3289 4016	The Sum of theſe Parts is,	12952
	2900 5007 1606	The Sum of theſe, is	9513
The total Sum of all theſe Parts	22465	The Sum of each Parcel put together	22465

Sect. 3. *Of* Subtraction.

Poſtulate or Petition.

That any Number *may be Diminiſhed, or made Leſs, by taking another* Number *from it.*

SUBTRACTION is that Rule by which one Number is deducted or taken out of another, that ſo the Remainder, Difference, or Exceſs may be known.

As *6* taken out of *9*, there remains 3. This 3 is alſo the Difference betwixt *6* and *9*, or it is the Exceſs of *9* above *6*.

Therefore the Number (or Sum) out of which Subtraction is required to be made, muſt be greater than (or at leaſt equal to) the Subtrahend or Number to be Subtracted.

Note, *This Rule is the Converſe or direct Contrary to* Addition.

And here the ſame Caution that was given in Addition; of placing Figures directly under thoſe of the ſame Value, *viz.* Units under Units, Tens under Tens, and Hundreds under Hundreds, *&c.* muſt be carefully obſerved; alſo underneath the loweſt Rank there muſt be drawn a Line (as before in Addition) to ſeparate the given Numbers from their Difference when it is found.

Then having placed the leſſer Number under the greater, the Operation may be thus performed.

RULE.

Begin at that Right-Hand Figure or Place of Units (as in Addition) and take or ſubtract the lower Figure in that Place from the Figure that ſtands over it, ſetting down the Remainder

or Difference underneath its own Place, if the two Figures chance to be equal, ſet down a Cypher. But if the upper Figure be leſs than the lower Figure, then you muſt add 10 to the upper Figure, or mentally call it 10 more than it is, and from that Sum ſubtract the lower Figure, ſetting down the Remainder (as before directed.) Now becauſe the 10 thus added, was ſuppos'd to be borrow'd from the next ſuperior Place (*viz.* of Tens) in the upper Figures, therefore you muſt either call the upper Figure in that Place from whence the 10 was borrow'd, one leſs than really it is, or elſe (which is all one, and moſt uſual) you muſt call the lower Figure in that Place one more than it really is, and then proceed to Subtraction in that Place, as in the Former; and ſo gradually on from one Row of Figures to another until all be done.

Example 1.

Let it be required to find the Difference between 6785, and 4572. That is, let 4572 be ſubtracted from 6785.

Theſe Numbers being placed down, as before directed, will ſtand

$$\text{Thus}\left\{\begin{array}{r}6785\\4572\\\hline 2213\end{array}\right.$$

Beginning at the Place of Units, take 2 from 5 and there will remain 3 which muſt be ſet down underneath its own Place, and then proceed to the Place of Tens, taking 7 from 8, there will remain 1, to be ſet down underneath its own Place; again at the Place of Hundreds take 5 from 7, and there remains 2, which ſet down, as before; laſtly, take 4 from 6 and there will remain 2, which being ſet down underneath its own Place, the Work is finiſhed, and the Difference ſo found will be $2213 = 6785 - 4572$, as was required.

Example 2.

The Difference between 5849, and 7496 is required.

Having placed the Numbers as in the Margin, begin at the Place of Units (as before) and ſay 9 from 6 cannot be, but 9 from 16 and there remains 7, to be ſet down under its own Place, next proceed to the Place of Tens, where you muſt now pay the 10 that was borrowed to make the 6, 16, by accompting the upper Figure 9 in that Place one leſs than it is, ſaying, 4 from 8 and there remains 4, or elſe (which is the moſt practiſed) ſay 1 I borrowed and 4 is 5 from

$$\begin{array}{r}7496\\5849\\\hline 1647\end{array}$$

from 9 and there Remains 4, to be ſet down under it's own Place (as before); again at the Place of Hundreds, ſay 8 from 4 that cannot be, but 8 from 14 there will remain 6 to be ſet down; and here I have borrowed 10 (as before) which muſt be paid in the ſame Manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, ſaying, 5 from 6 there remains 1, or elſe by ſaying 1 borrowed and 5 is 6 from 7 and there remains 1, which being ſet down under its own Place all is done, and the Difference required will be 1647=7496—5849.

Example 3.

From 830476
Take 741068

Remains 89408

By this Example you may perceive that Cyphers in the Subtrahend, *viz.* in the Numbers to be ſubtracted, do not diminiſh the Number from whence Subtraction is made. See *Page* 4.

Theſe three Examples, I preſume, may be ſufficient to ſhew the young Learner the Method of Subtracting whole Numbers; as for the Reaſon thereof it is the ſame with that of Addition, Page 10, *viz.* of *the Whole being equal to all its Parts taken together.*

That is, in this Rule the Number from which Subtraction is required to be made, is underſtood to be the Whole, and the Subtrahend or Number to be ſubtracted, is ſuppos'd to be a Part of that Whole, conſequently if that Part be taken from the Whole the Remainder will be the other Part.

From hence is deduced the common Method of proving Subtraction, by adding together the Subtrahend and the Remainder. For if the Sum of thoſe Two which are here called Parts, be equal to the Number from whence Subtraction was made (which is here called the whole) then the Work is right; if not, care muſt be taken to diſcover and correct the Error.

Example.

From 59435
Take 47608 } Add
11827 }

Proof { 59435 } The Sum which is equal to the Number from whence Subtraction was made.

Or

Or from the abovesaid Reason, it will be easy to conceive how to prove the Truth of Subtraction by Subtraction.

For if from	59435	being here the Whole,
there be taken	47608	as Part of that Whole,
there will remain	11827	the other Part (as before)
And if from	59435	the Whole, there be subtracted
the last Part, *viz.*	11827	
there will remain	47608	the first Part, or Number which was required to be first subtracted.

From	75643	From	7000000
Take	9000	Take	986432
	66643	Remains	6013568

Sect. 4. *Of* Multiplication.

MULTIPLICATION is a Rule by which any given Number may be speedily increased, according to any proposed Number of Times.

That is, One Number is said to multiply another, when the Number multiplied is so often added to it self, as there are Units in the Number multiplying; and another Number is produced. (*Euclid.* 7. *Def.* 15.)

To perform *Multiplication* there are required two given Numbers, called *Factors.*

The First is that Number which is to be multiplied, and is generally put the greatest of the two Numbers, commonly call'd the Multiplicand.

The other is that Number by which the First is to be multiplied, and is usually called the Multiplicator or Multiplier; and this denotes the Number of Times that the Multiplicand is required to be added to it self. For so many Units as are contained in the Multiplier, so many times will the Multiplicand be really added to it self, (as *per Euclid* above.) And from thence will arise a third Number, called the *Product.* But in *Geometrical Operations* it is called the *Rectangle* or *Plane.*

For Instance; Suppose it were required to increase 6 four Times, That is, to multiply 6 into or with 4, these two Numbers are to be set (or placed) down as in *Addition* or *Subtraction.*

Thus

Thus $\left\{\begin{array}{ll} 6 & \textit{Multiplicand,} \\ 4 & \textit{Multiplier,} \end{array}\right\}$ or *Factors.*

Product 24 *viz.* 4 times 6 is 24, as plainly appears by *Addition, viz.* By ſetting down 6 four times, and then adding them together into one Sum Thus

1	6	Add
2	6	
3	6	
4	6	
	24	

From hence it is evident that *Multiplication* is only a conciſe or compendious Way of adding any given Number to it ſelf, ſo often as any Number of Times may be propoſed.

Before any Operation can be readily perform'd in *Multiplication*, the ſeveral Products of the ſingle Figures, one into another muſt be perfectly learn'd by Heart, *viz.* That 2 times 2 is 4, that 3 times 3 is 9, that 3 times 6 is 18, *&c.* According as they are expreſſed in the following *Table.* Wherein I have omitted multiplying with 2, it being ſo very eaſy that any one may do it.

Multiplication Table.

3×3=9	4×4=16	5×5=25	6×6=36	7×7=49	8×8=64
3×4=12	4×5=20	5×6=30	6×7=42	7×8=56	8×9=72
3×5=15	4×6=24	5×7=35	6×8=48	7×9=63	9×9=81
3×6=18	4×7=28	5×8=40	6×9=54		
3×7=21	4×8=32	5×9=45			
3×8=24	4×9=36				
3×9=27					

I think it needleſs to give any Explanation of this *Table*; for if the Signs and their Significations be well underſtood, (*vide Page* 5) it muſt needs be eaſy. Only this may be noted, that 4×3=3×4, or 7×5=5×7, *&c.*

That is, 3 times 4, is the ſame with 4 times 3, or 5 times 7, is the ſame with 7 times 5, *&c.* The like muſt be underſtood of all the reſt in the *Table.*

And when all theſe ſingle Products are ſo perfectly learn'd by Heart, as to be ſaid without pauſing; you may then proceed (but not till then) to the Buſineſs of *Multiplication*; which will be found very eaſy, if the following *Rule* (and *Examples*) be carefully obſerved.

RULE.

Always begin with that Figure which ſtands in the Units Place of the Multiplier; and with it multiply the Figure which ſtands in

in the Units Place of the Multiplicand; if their Product be less than Ten, set it down underneath its own Place of Units, and proceed to the next Figure of the Multiplicand. But if their Product be above Ten (or Tens) then set down the Overplus only (or odd Figure, as in Addition) and bear (or carry) the said Ten or Tens in mind until you have multiplied the next Figure of the Multiplicand, with the same Figure of the Multiplier; then to their Product add the Ten or Tens carried in Mind, setting down the Overplus of their Sum above the Tens, as before: And so proceed on in the very same Manner, until all the Figures of the Multiplicand are multiplied with that Figure of the Multiplier.

Example I.

Suppose it were required to multiply 3213 into or with 3.

$$\left.\begin{array}{r}3213 \ \textit{Multiplicand,} \\ 3 \ \textit{Multiplier,}\end{array}\right\}\text{or } \textit{Factors.}$$

Product 9639

Beginning at the Units Place, say, 3 times 3 is 9, which because it is less than Ten, set it down underneath its own Place, and proceed to the next Place of Tens; saying 3 times 1 is 3, which set down underneath its own Place, then to the next Place, *viz.* of Hundreds, saying 3 times 2 is 6, which set down, as before; Lastly, at the Place of Thousands, say 3 times 3 is 9, which being set down underneath its own Place, the Operation is finished; and the true Product is $9639 = 3213 \times 3$, as was required.

Example 2.

Let it be required to multiply 8569 into 8. Set down these Numbers as before,

$$\text{Thus}\left\{\begin{array}{r}8569 \\ 8 \\ \hline 68552\end{array}\right.$$

Beginning at the Units Place say, 8 times 9 is 72, set down the 2 underneath its own Place of Units, and bear the 70, or 7 Tens in mind, and proceed to the next Figure of the Multiplicand (at which Place the 7 Tens are only 7) saying 8 times 6 is 48, and the 7 carried in mind is 55, set down the odd 5 underneath its own Place of Tens, and carry the 50 (which is really 500) to the next Place (*viz.* of Hundreds) at which Place it is only 5, where say, 8 times 5 is 40; and the 5 carried in mind is 45, set down the 5 underneath its own Place; and carry the 40 or 4 Tens (which is really 4000) to the next Place, *viz.* of Thousands,

Thouſands, ſaying, 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the laſt Place or Figure to be multiplied) ſet down the whole Product 68, and the Work is done.

So that, $8569 \times 8 = 68552$, the Product required.

Now the Reaſon of this and all other the like Operations, may be eaſily conceived from this which follows,

8569 8	The ſame Factors as before,
72	Here 8 times 9 is but 72, as before, becauſe the 9 ſtands in the Units Place.
480	Now here it is not really 8 times 6=48, but it is 8 times 60=480, becauſe the 6 ſtands in the Place of Tens.
4000	And here it is not 8 times 5=40, but it is really 8 times 500=4000, becauſe the 5 ſtands in the Place of Hundreds.
64000	Laſtly, becauſe the 8 in the Multiplicand ſtands in the Place of Thouſands, it is therefore, 8 times 8000=64000, and not 8 times 8=64.
68552	The Sum of the particular Products, which gives the true Product, as before.

By what hath been already ſaid, with a little Conſideration had to the Examples: I preſume the Learner may eaſily underſtand how to multiply whole Numbers with any ſingle Figure. And when it is requir'd to multiply with more than one; Then ſo many Figures as there are in the Multiplier, ſo many particular Products there muſt be.

That is, all the Figures of the Multiplicand muſt be multiplied with every ſingle Figure of the Multiplier as if there were but one ſingle Figure; and the Sum of all thoſe particular Products, will be the true Product required; but in thoſe Operations great Care muſt be taken in ſetting down the particular Products, (which ariſe by each multiplying Figure) in their proper Places: Which will be eaſily done if the following Directions be carefully obſerv'd.

Viz. Always place the firſt Figure (or Cypher) of every particular Product, directly underneath the multiplying Figure. Or thus:

The firſt Figure (or Cypher) of the ſecond particular Product muſt ſtand directly under the ſecond Figure (or Place) of the firſt Product; and the firſt Figure (or Cypher) of the third

particular Product, must stand directly underneath the third Figure of the first Product: And so on until all is done.

Now the Reason of placing the first Figure of every particular Product in this Order, will be very obvious to any one that considers the last Example; wherein the Cyphers are only set down to shew the true distance of the first Figure in each particular Product from the Units Place. And altho' it is not usual to set down Cyphers in this Manner; yet they are always suppos'd to be there: That is, their Places are always left void, as in the two following Examples; wherein I have placed Points instead of Cyphers.

Example 3.

Let it be required to multiply 78094, into or with 7563.

```
   78094 } Factors.
    7563 }
  ------
  234282     The First particular Product with           3
  468564.    The Second particular Product with         60
 390470..    The Third particular Product with         500
546658...    The Fourth particular Product with       7000
---------
590624922    The Total, or true Product required.
```

Example 4.

Suppose it be required to multiply 57498 into 60008.

```
     57498
     60008
    ------
    459984     The Product with         8
344988....     The Product with     60000
----------
3450339984=57498×60008 as was required.
```

Here you may observe, that I pass over the Cyphers, and only take care of placing the first Product of the last Figure, *viz.* of 60000 according to the foregoing Directions.

When there is a Cypher or Cyphers, to the Right-hand either of the Multiplicand, or Multiplicator, or to both; in that Case multiply the Figures as before; neglecting the Cyphers until the particular Products are added together; then to their Sum annex so many Cyphers as are in either or both the Factors. As in these.

Example 5.

Example 5.

```
  9538
  4600
  ----
 57228
38152
--------
43874800
```

Example 6.

```
  87600
     79
  -----
  7884
 6132
-------
6920400
```

Example 7.

```
     785000
      56900
     ------
     7065
    4710
   3925
-----------
44666500000
```

Take a few Examples without their Work at large.

75649×579=43800771
687000×356=244572000
530674×45007=23884044718
7901375×30000=237041250000
537084000×590700=317255518800000
102030405×504030201=51426405540261405
987654321×123456789=121932631112635269

Note, If it be required to multiply any Number with 10, 100, 1000, 10000 *&c.* it is only annexing the Cyphers of the Multiplier to the Figures of the Multiplicand, and the Work is done.

Thus { 578×10 =5780. 578×1000 =578000.
578×100=57800. 578×10000=5780000, *&c.*

Theſe Examples (being well underſtood) are ſufficient to inſtruct the Learner, in all the Varieties that can happen in Multiplying of whole Numbers, according to the Method generally practiſed: However it may not be amiſs to ſhew here how Multiplication may be performed (with many Figures) by Addition only.

Example.

Let it be required to multiply 879654 into 79863.
In Order to perform this (or any other Operation of this Kind) by Addition only; you muſt make a Tariffa or ſmall Table of the given Multiplicand, in this Manner:
Firſt, Make a ſmall Column, and in it place gradually downward the nine ſingle Figures; *viz.* 1. 2. 3. 4. 5. *&c.*

Then againſt the Figure 1, ſet down the Multiplicand (which in this Example is 879654) and againſt the Figure 2, ſet down the Double of the Multiplicand, found by adding it to it ſelf; To this Double add the Multiplicand, ſetting down their Sum againſt the Figure 3. And ſo proceed on by a continued Addition until there be ten Times the Multiplicand in the Table, which, if the Work is true, will be the Multiplicand it ſelf with a Cypher to the Right-hand of it (as in the annexed Table) this being done, it will be eaſy to conceive, that the Figures in the ſmall Column of the Table, do reſpectively repreſent thoſe of the Multiplier; And that the Numbers againſt any of thoſe Figures in the ſmall Column, will be the true Product of the Multiplicand agreeing to any Figure of the Multiplier; as plainly appears by the Work of this Example.

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then	879654 79863	} The *Factors* as before.
Againſt 3, in the Table, is	2638962	=879654×3
Againſt 6, is	5277924	=879654×60
Againſt 8, is	7037232	=879654×800
Againſt 9, is	7916886	=879654×9000
Againſt 7, is	6157578	=879654×70000
The Product required.	70251807402	=879654×79863

Note, This Method of Tabulating the Multiplicand, is both eaſy and certain; being neither ſubject to Errors, nor burdenſom to the Memory, and therefore in large Calculations it may be found very uſeful. But for common Practice the uſual Method (as in *Page* 18, *&c.*) is beſt, and to be preferr'd before this.

Moſt Maſters that teach (and ſeveral Authors that write of) Arithmetick, do teach to prove the Truth of Multiplication, by caſting away all the Nines that are contain'd in both the Factors, and their Product; but becauſe that Method is very erroneous, as might be eaſily ſhew'd; I ſhall therefore omit inſerting it, and leave the Proof of Multiplication to the next *Section*, wherein (I preſume) the Reaſon and Proof, both of it, and Diviſion, will plainly appear.

Sect. 5. *Of* Division.

Division is a *Rule* by which one *Number* may be ſpeedily *ſubtracted* from another, ſo many times as it is contained therein.

That is, It ſpeedily diſcovers how often one Number is contained (or may be found) in another: And to perform that, there are required two Numbers to be given.

1. The one of them is that Number which is propoſed to be divided, and is called the *Dividend*.

2. The other is that Number by which the ſaid *Dividend* is to be divided, and is called the *Diviſor*.

And by comparing theſe Two, *viz.* the Dividend and the Diviſor together, there will ariſe a third Number, called the *Quotient*; which ſhews how often the Diviſor is contained in the Dividend, or into what Number of equal Parts the Dividend is then divided. Therefore,

Diviſion is by *Euclid* fitly term'd the meaſuring of one Number by another, *viz.* one Number is ſaid to meaſure another by that Number, which when it multiplies, or is multiplied by it, it produceth. *Euclid.* 7. *Def.* 23.

And, if a Number meaſuring another multiply that Number by which it meaſureth, or be multiplied by it, it produceth the Number which it meaſureth. *Euclid.* 7. *Axiom* 9.

That is to ſay, If that Number which divides another, (called the Diviſor) be multiplied with the Number which is produced by Diviſion (called the Quotient) their Product will be the Number divided or Dividend. Whence it follows that *Diviſion* and *Multiplication* are the Converſe or direct Contrary one to another (as *Subtraction* is to *Addition*) and do mutually prove the Truth of each others Operations.

I ſhall therefore make Choice of the foregoing Examples in *Multiplication*, in order (as I preſume) to render the Buſineſs of *Diviſion* more plain and eaſy.

Firſt, Let it be required to find how often 6 is contained in 24. That is, to divide 24 by 6.

N. B. Always place down the given Numbers in this Order; Firſt ſet down the *Diviſor*, and to the Right-hand of it draw a crooked Line; then ſet down the *Dividend*, and to the Right of it draw another crooked Line, in which muſt be placed the *Quotient* Figure, or Figures as they are found.

Dividend

Dividend
Thus, *Divisor* 6) 24 (4 the *Quotient.*

Here I consider how many times 6 there is in 24, and find it 4, *viz.* 4 times 6 is 24, therefore 4 is the true Quotient or Answer required.

This is apparent by Subtraction, as in the Margin, where 24 the Dividend being set down, and from it 6, the Divisor is continually subtracted so often as it can be, which is just 4 times. Therefore 4 is the true Quotient or Answer required.

Compare this with the *Example, Page* 15.

	24
1	6
—	—
	18
2	6
—	—
	12
3	6
—	—
	6
4	6
	0

COROLLARY.

From hence it is evident; that *Division* is but a concise or compendious Method of subtracting one Number from another so often as it can be found therein; for if the Divisor be continually subtracted from the Dividend, accounting an Unit (or 1) for each time it is subtracted (as above) the Sum of those Units will be the Quotient.

All Operations in *Division* do begin contrary to those of *Multiplication, viz.* at the first Figure to the Left-hand, or that of the highest Value, and decrease the Dividend by a repeated Subtraction of each Product arising from the Divisor when multiplied into the Quotient Figure. And the only Difficulty in *Division* of whole *Numbers* (or indeed of any *Numbers*) lies in making Choice of such a Quotient Figure, as is neither too big, nor too little; and that may be easily obtained by observing the following *Rule,* which hath two *Cases.*

RULE.

Case 1. As often as the first Figure of the Divisor is taken from the first Figure of the Dividend: So often must the second Figure of the Divisor be taken from the second Figure of the Dividend, when it is joined with what remains of the First. And as often must the third Figure of the Divisor be taken from the third Figure of the Dividend, &c.

But if the first Figure of the Divisor cannot be taken from the first Figure of the Dividend. Then,

Case

Case 2. So often as the first Figure of the Divisor, is taken from the two first Figures of the Dividend, so often must the second Figure of the Divisor be taken from the third Figure of the Dividend, when it is join'd with what remain'd of the Second: And so often must the third Figure of the Divisor be taken from the fourth Figure of the Dividend, *&c.*

That is, the Quotient Figure must be such, as being multiplied into the Divisor, will produce a Product equal to such a Part of the Dividend as is then taken for that Operation: But if such a Product cannot be exactly found, then the next less must be taken, and ordered, as in the following Examples: Of which let that in *Page 16* be the first, wherein there was given 8569 the Multiplicand, and 8 the Multiplier. To find the Product 68552. Let us here suppose the said Product 68552, and 8 the Multiplier, both given; thence to find the Multiplicand. That is, Let it be required to divide 68552 by 8.

Dividend
Divisor 8) 68552 (*Quotient* when found.

According to the *Rule, Case* 1. I compare 8 the Divisor with 6 the first Figure of the Dividend, and finding I cannot take it from that; I then consider (by *Case* 2.) how often 8 can be taken from 68, the two first Figures of the Dividend, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest Product of 8 (into any Figure) that can be taken from 68. I therefore place 8 in the Quotient, and with it multiply 8 the Divisor, setting down their Product underneath the said two first Figures of the Dividend, subtracting it from them, and then the Work will stand

Thus 8) 68552 (8
64
——
4

In order to a second Operation, I make a Point under the next Figure of the Dividend, *viz.* under the 5, and bring it down underneath its own Place to the Remainder 4, which will by that Means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the Quotient, and with it multiply 8 the Divisor, setting down and subtracting their Product, as before. Then the Work will stand

Thus

```
Thus  8) 68552 (85
         64·
         ---
          45
          40
          --
           5
```

For a third Operation, I make a Point under the next Figure of the Dividend *viz.* under the 5, and bring it down, as before, proceeding in all Reſpects, as before; and then the Work will ſtand

```
Thus  8) 68552 (856
         64··
         ---
          45
          40
          --
           55
           48
           --
            7
```

Laſtly, I point and bring down the 2, *viz.* the laſt Figure of the Dividend to the Remainder 7, which will then become 72, and proceeding as in the other Operations, I find that 8, the Diviſor, can be taken juſt 9 times from 72, and the Work is finiſhed, and will ſtand

```
Thus  8) 68552 (8569
         64···
         ---
          45
          40
          --
           55
           48
           --
            72
            72
            --
            (0)
```

The true Quotient is found to be 8569, being exactly the eighth Part of 68552, or the Multiplicand of the propoſed Example of Multiplication. As was required.

The Reaſon of theſe Operations will be very plain to any one that will a little conſider of it as follows.

Diviſor

Divisor 8) 68552 (8000. The First Quotient or Figure.

Subtract 64000 — This Product of the Divisor into the Quotient is 64000. *viz.* 8 Times 8000. the Quotient Figure being always of the same Value or Degree with that Figure under which the Unit's Place of its Product stands.

Divisor 8) 4552 (500. The second Quotient Figure.

Subtract 4000 — And here the Product is 4000. *viz.* 8 times 500. not 8 times 5, *&c.*

Divisor 8) 552 (60. The third Quotient Figure.

Subtract 480 — Also here the Product is 480. *viz.* 8 times 60. for the Reasons abovesaid.

Divisor 8) 72 (9. The fourth Quotient Figure.

Subtract 72 — Now here the Product is but 72. *viz.* 9 times 8. because the 9 stands in the Place of Units.

Remains (00) Now the Sum of all the several Quotients, *viz.* 8000 X 500 X 60 X 9=8569. as before.

If the Procefs of this Example be well confidered and compared with that of Multiplication, *Page* 17. it will evidently appear to be only the Converfe of that; for the particular Products are alike in both, only that which is *last* there, is *first* here; there they are *added*, here they are *fubtracted*. So that whoever underftands the true Reafon of the one, muft needs underftand the Reafon of the other, and then Divifion will become very eafie, although the Divifor confift of feveral Places of Figures.

Example.

Let it be required to divide 590624922 by 7563.

Dividend

Divifor 7563) 590624922 (

'Tis plain at Sight, that 7563 the *Divifor*, cannot be taken from 5906. the like Number of Figures in the *Dividend*.

Therefore, by the fecond Cafe of the Rule (*Page* 23.) there muft be allowed Five Figures of the Dividend, *viz.* 59062 for the firft Operation or Quotient; that fo the firft Figure 7 of the Divifor may be taken out of the two firft Figures, *viz.* 59 of the Dividend, *&c.*

Then I proceed (*per Case* 2.) and consider how often 7 may be taken from 59. and find it may be taken 8 times, for 8 times 7 is but 56. which I mentally subtract from 59. and there remains 3; to this 3 I mentally adjoyn the third Figure of the Dividend, *viz.* 0. which makes it 30. out of which I must take the second Figure of the Divisor, *viz.* 5. so often as I took the 7 from 59. which was 8 times. But that cannot be, for 8 times 5 is 40. which is more than 30. therefore 8 is too big a Figure to be placed in the Quotient; Yet, hence I conclude, that the next Less, *viz.* 7 may be taken without any further Tryal. I therefore place 7 in the Quotient, and with it multiply the Divisor, setting down their Product under the Dividend, and subtract it from thence, as in the other Example, and then the Work will stand

```
Thus 7563) 590624922 (7
           52941
           -----
            6121
```

In order to a second Operation, I make a Point under the next Figure of the Dividend, *viz.* under the 4. and bring it down to the Remainder 6121, which will then become 61214. with which I proceed in all respects as I did before with the 59062. and find the next Quotient Figure will be 8. with which I multiply the Divisor, &c. and subtract their Product from the said 61214. Then the Work will stand

```
Thus 7563) 590624922 (78
           52941
           -----
            61214
            60504
            -----
              710
```

To this Remainder 710. I point and bring down the next Figure of the Dividend, *viz.* 9. which makes it 7109; now because the Divisor 7563 cannot be taken from 7109. I therefore place a Cypher in the Quotient.

And this must always be carefully observed, *viz.* That for every Figure or Cypher, which is brought down from the Dividend, in order to a new Operation, there must always be either a Figure or Cypher, set down in the Quotient. Then the Work will stand

Thus

Thus 7563) 590624922 (780
52941··
61214
60504
7109

To this 7109, I bring down another Figure of the Dividend, *viz.* 2, and then it will become 71092, then I consider how often 7 can be taken from 71, &c. (just as at the first Operation) and find it may be taken 9 times, therefore I set down 9 in the Quotient, and with it multiply the Divisor, setting down and subtracting their Product, as before; Then the Work will stand

Thus 7563) 590624922 (7809
52941···
61214
60504
71092
68067
3025

To this Remainder 3025, I point and bring down the last Figure 2 of the Dividend, which makes it 30252, then proceeding in all Respects as before, I find the Quotient Figure to be 4, with it I multiply the Divisor, setting down and subtracting their Product as before, and then the Work will stand

Thus 7563) 590624922 (78094
52941····
61214
60504
71092
68067
30252
30252
(00000)

Here the Work is ended, and I find the Quotient to be 78094, being the true Multiplicand of the propos'd Example of Multiplication, *Page* 18.

That is, 7563 is contained in 590624922, just 78094 times, &c.

If the Work of this Example be conſidered and compared with the Rule (*Page* 22.) the whole Buſineſs of Diviſion will be eaſie; for indeed the only Difficulty (as I ſaid before) lies in making choice of a true Quotient Figure, which cannot well be done according to the common Method of Diviſion, without Trials, yet thoſe Trials need not be made with the whole Diviſor (as appears by this laſt Example) for by the two firſt Figures of the Diviſor all the reſt are generally regulated; except the ſecond Figure chance to be 2, 3, or 4, and at the ſame time the third Figure be 7, 8, or 9, then indeed Reſpect muſt be had to the third Figure according as the Rule directs.

However, if thoſe Trials are thought too troubleſom, they may be avoided, and the ſame Quotient Figures may both eaſily and certainly be found by Help of ſuch a ſmall Table made of the Diviſor, as was of the Multiplicand in *Page* 20.

Example 4.

Let it be required to divide 702518074O2 by 79863. *See the Example of Multiplication,* Page 20, *and as there directed, make a Table of the Diviſor* 79863.

Thus,

	Diviſor	*Dividend*	
1	79863)	702518074O2	(879654. *Quotient.*
2	159726	638904.....	
3	239589	————	
4	319452	636140	
5	399315	559041	
6	479178	————	
7	559041	770997	
8	638904	718767	
9	718767	————	
	————	522304	
10	798630	479178	
		————	
		431260	
		399315	
		————	
		319452	
		319452	
		————	
		(000000)	

The Work of this Operation I preſume may be eaſily underſtood. For thoſe Figures in the Table are the Products of the Diviſor into all the 9 Figures; conſequently thoſe Figures in the ſmall Column do ſhew what Figure is to be placed in the Quotient; without any doubtful Trials of the Diviſor with the Dividend, as before.

This Method of tabulating the Diviſor, may be of good Uſe to a Learner. Eſpecially until he is well practiſed in Diviſion; yea, and even then if the Diviſor be large, and a Quotient of many Figures be required; as in reſolving of high Æquations, and calculating of Aſtronomical Tables, or thoſe of Intereſt, &c.

Hitherto

Hitherto I have made choice of Examples, wherein the Dividend is truly meaſured or divided off, by the Diviſor, without leaving any Remainder, being thoſe as were compoſed of the Diviſor and Quotient. But it moſt uſually falls out, that the Diviſor will not exactly meaſure the Dividend; in that caſe the Remainder (after Diviſion is ended) muſt be ſet over the Diviſor with a ſmall Line betwixt them adjoyning to the Quotient.

Example 5.

Suppoſe it were required to divide 379 by 5.

$$
\begin{array}{l}
5)\ 379\ (75\frac{4}{5}\ \text{the *Remainder*, the *Diviſor*.} \\
\quad 35 \\
\hline
\quad\ \ 29 \\
\quad\ \ 25 \\
\hline
\textit{Remains}\ (4)
\end{array}
$$

Example 6.

Again, Let it be required to divide 43789 by 67.

$$
\begin{array}{l}
67)\ 43789\ (653\frac{38}{67}\ \text{The true Quotient required.} \\
\quad 402 \\
\hline
\quad\ \ 358 \\
\quad\ \ 335 \\
\hline
\quad\ \ \ 239 \\
\quad\ \ \ 201 \\
\hline
\textit{Remains}\ (38)
\end{array}
$$

How ſuch Remainders thus placed over their Diviſors (which are indeed Vulgar Fractions) may be otherwiſe managed, ſhall be ſhew'd farther on.

N. B. When the Diviſor happens to be an Unit, *viz.* 1. with a Cypher, or Cyphers annexed to it, As 10, 100, 1000, *&c.* Diviſion is truly performed by cutting off with a Point or Comma, ſo many Figures of the Dividend as there are Cyphers in the Diviſor; then are thoſe Figures ſo cut off to be accounted a Remainder, and the reſt of the Figures in the Dividend will be the true Quotient required, becauſe an Unit or 1 doth neither multiply nor divide.

Example 7.

Let it be required to divide 57842 by 100. The Work may ſtand thus, 100) 578.42 the Quotient required, or thus 100) 57842 ($578\frac{42}{100}$ the ſame as before.

Hence it follows, that if any Diviſor have Cyphers to the Right-hand of it, you may cut off ſo many of the laſt Figures in

in the Dividend, and divide the other Figures of the Dividend, by those Figures of the Divisor that are left when the Cyphers are omitted. But when Division is ended, those Cyphers so omitted in the Divisor, and the Figures cut off in the Dividend are both to be restored to their own Places.

Example 8.

Suppose it were required to divide 675469 by 5400.

5400) 675469 (125
54..
135
108
274
270

Remains (4) But the true Remainder is 469.
Consequently the true Quotient is 125$\frac{469}{5400}$.

As to the Manner of proving the Truth of any Operation, either in Multiplication, or Division, I presume it may be easily understood, by what is deliver'd in *Page* 21, compared with the three first Examples of Division; For from thence it will be easie to conceive, that if the Divisor and Quotient be multiplied together, their Product (with what remains after Division being added to that Product) will be equal to the Dividend. As in the fifth Example, wherein the Dividend is 379, the Divisor is 5, the Quotient is 75, and the Remainder is 4.

I say, $75 \times 5 = 375$, to which add the Remainder 4, it will be 379.

Again, in the sixth Example, the Divisor is 67, the Quotient is 653, and the Remainder is 38.

Then $653 \times 67 = 43751$, and $43751 + 38 = 43789$ the Dividend, *&c.*

There are several useful Contractions, both in Division and Multiplication, which I have purposely omitted until I come to treat of Decimal Arithmetick. Also I have omitted the Business of Evolution or Extracting of Roots, until further on; and so shall conclude this Chapter with a few Examples of Division, unwrought at large, leaving them for the Learner's Practice.

579) 43800771 (75649.
Or 75649) 43800771 (579

45007)

45007) 23884044718 (530674.
Or 530674) 23884044718 (45007.
356) 244572000 (687000.
59600) 57659066400 (967434.
10000) 679543820000 (67954382.
79) 282016 (3569$\frac{65}{79}$.

CHAP. III.

Concerning Addition *and* Subtraction *of* Numbers *of different* Denominations, *and how to* reduce *them from one* Denomination *to another.*

SECT. I.

1. *Of* English Coin.

THE least Piece of Money used in *England* is a Farthing, and from thence ariseth the rest, as in this Table.

Farth.
4 = 1 *d. Pen.*
48 = 12 = 1 *s. Shil.*
960 = 240 = 20 = 1 *l. Pound Sterling.*

5 *s.* is a Crown.
10 *s.* is an Angel.
6 *s.* 8 *d.* a Noble.
13 *s.* 4 *d.* a Mark.

Note, When *l. s. d. q.* are placed over, (or to the Right-hand of) Numbers, they denote those Numbers to signify Pounds, Shillings, Pence, Farthings.

	l.	*s.*	*d.*	*q.*
As	35	10	6	2.

Or 35 *l.* 10 *s.* 6$\frac{1}{2}$ *d.* Either of these do signify 35 Pounds, 10 Shillings, 6 Pence, 2 Farthings.

The same must be understood of all the following Characters, belonging to their respective Tables, *viz.* Of Weights, Measures, *&c.*

2. *Troy Weight.*

The Original of all Weights used in *England*, was a Corn of Wheat gathered out of the Middle of the Ear, and being well dried, 32 of them were to make one Penny Weight, 20 Penny Weight

Weight one Ounce, and 12 Ounces one Pound Troy. *Vide Statutes of* 51. *Hen.* 3. 31. *Edw.* 1. 12. *Hen.* 7.

But in later Times it was thought sufficient to divide the aforesaid Penny Weight into 24 equal Parts, called Grains, being the least Weight now in common Use; and from thence the rest are computed as in this Table.

Gr. Gra.			
24 = 1 *P. W. Penny Weight.*			
480 =	20 = 1 *Oz. Ounce.*		
7560 =	240 =	12 =	1 ℔ *Pound.*

Note, By Troy Weight are weighed Jewels, Gold, Silver, Corn, Bread and all Liquors.

Besides the common Divisions of Troy Weight, I find in *Angliæ Notitia*, or, *The Present State of* England, Printed in the Year 1699, *That the Moneyers*, (as that Author calls them) *do subdivide the Grain.*

The
- 24 *Blanks* = 1 *Periot.*
- 20 *Periots* = 1 *Droite.*
- 24 *Droites* = 1 *Mite.*
- 20 *Mites* = 1 *Grain*, &c. as before.

3. *Apothecaries Weights.*

The Apothecaries divide a Pound Troy, as in this Table.

Gr. Grain.				
20 = 1 ℈ *Scruple.*				
60 =	3 = 1 ʒ *Dram.*			
480 =	24 =	8 = 1 ℥ *Ounce.*		
5760 =	288 =	96 =	12 =	1 ℔ *Troy*, the same as before.

By these Weights the Apothecaries compound their Medicines, but buy and sell their Drugs, by *Averdupois Weight.*

4. *Averdupois Weight.*

When *Averdupois Weight* became first in Use, or by what Law it was first settled, I cannot find out in the Statute Books; but on the contrary, I find that there should be but one Weight (and one Measure) used throughout this Realm, *viz.* that of *Troy*, (*Vide* 14. *Ed.* 3. and 17. *Ed.* 3.) So that it seems (*to me*) to be first introduced by Chance, and settled by Custom, *viz.* from giving good or large Weight to those Commodities as are usually weigh'd by it, which are such as are either very course and drossy, or very

very subject to wast: As, all kind of Grocery Wares. And *Pitch, Tar, Rosin, Wax, Tallow, Soap, Flax, Hemp*, &c. *Copper, Tin, Steel, Iron Lead*, &c. Also, *Flesh, Butter, Cheese, Salt*, &c. To these and the like (I presume) it was thought convenient to allow a greater Weight than what the Laws had provided, which happened to be about a sixth Part more: For I found by a very nice Experiment, that one Pound *Averdupois* is equal to 14 *Ounces*, 11 *Penny-weight*, and $15\frac{1}{2}$ *Grains Troy*. And it is now computed as in the following Table.

Drams.	*oz. Ounces.*	*lb Pounds.*	*C. Hundred.*	*Tun.*
16 =	1			
256 =	16 =	1		
28672 =	1792 =	112 =	1	
573440 =	35840 =	2240 =	20 =	1

And, lb:
- 14 = *a Stone.*
- 28 = $\frac{1}{4}$ *of C.*
- 56 = $\frac{1}{2}$ *of C.*
- 84 = $\frac{3}{4}$ *of C.*

5. *Long Measure.*

As the least Part of Weight came at first from a Wheat Corn, so (it's generally said) the least Part of *Long Measure* was at first a Barly Corn, taken out of the Middle of the Ear, and being well dried three of them in length were to make one Inch; and thence the rest, as in this Table.

And:
- 4 *Nails* = $\frac{1}{4}$ of a *Yard.*
- $1\frac{1}{4}$ *Yard* = 1 *Ell.*
- 2 *Yards* = 1 *Fathom.*

Barly Corns	*In. Inches*	*F. Feet*	*Y. Yards*	*P. Poles*	*Furlong*	*Mile*
3 =	1					
36 =	12 =	1				
108 =	36 =	3 =	1			
594 =	198 =	$16\frac{1}{2}$ =	$5\frac{1}{2}$ =	1		
23760 =	7920 =	660 =	220 =	40 =	1	
190080 =	63360 =	5280 =	1760 =	320 =	8 =	1

Note, That forty Poles (or Perches) in Length, and Four in Breadth do make a Statute Acre of Land.

That is, 220 Yards, multiplied into 22 Yards = 4840 Square Yards are a Statute Acre.

And according to the Transactions of the *French Academy*, *Anno* 1687, a *Paris* Foot Royal is = 12.8 Inches *English*; Six of those Feet make a Toise; and 57060 Toise = 365184 *English* Feet, are the Measure of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 Miles and 288 Yards, which is very near to our Country-man Mr. *Norwood*'s Experiment made betwixt *London* and *York*, *Anno* 1635; who found that 367196 Feet = 69 Miles, and 858 Yards do make a Degree. And not 60 Miles, according to the common received Opinion and Practice of the Navigators or Seamen.

 Hence,

Hence, according to the *French* Account, the Circumference of the Earth (ſuppoſing it to be a true Spherical Figure) is 24899 *Engliſh* Miles.

6. Of *Liquid Meaſure.*

All Meaſures of Capacity, both Liquid and Dry, were at firſt made from *Troy Weight. Vide* Statutes *9. H.* 3. 51. *H.* 3. 12. *H.* 7. &c. wherein it is Enacted, that eight Pound *Troy Weight* of Wheat, gathered out of the Middle of the Ear, and well dried, ſhould make one Gallon of Wine Meaſure: And that there ſhould be but one Meaſure for Wine, Ale and Corn, throughout this Realm (*Vide* Stat. 14. *Ed.* 3. 15. *Ric.* 2.) But Time and Cuſtom hath alter'd Meaſures, as they have done Weights (and perhaps for one and the ſame Reaſon) for now we have three different Meaſures, *viz.* one for Wine, one for Ale or Beer, and one for Corn.

I have inſerted Tables of each as they are now computed by Cubic Inches, and practiſed in the Art of Gaging, *&c.*

The common Wine Gallon ſealed at *Guild-Hall* in *London*; by which all Wines, Brandies, Spirits, Strong-waters, Mead, Perry, Sider, Vinegar, Oyl and Honey, *&c.* are meaſured and ſold; is ſuppoſed to contain 231 Cubic Inches, and from thence the reſt are computed, as in this Table.

Cubic Inches

231 = 1 *G. Gallons*

9702 = 42 = 1 *Terce*

14553 = 63 = $1\frac{1}{2}$ = 1 *Hogſhead*

19404 = 84 = 2 = $1\frac{1}{3}$ = 1 *Puncion*

29106 = 126 = 3 = 2 = $1\frac{1}{2}$ = 1 *Butt or Pipe*

58212 = 252 = 6 = 4 = 3 = 2 = 1 *Tun*

Gallons.

Note, 18 = 1 *Runlet*, and $31\frac{1}{2}$ makes a *Wine* or *Vinegar Barrel.* (*Vide* 1. *R.* 3.)

But Doctor *Wybard* in his Tactometry, *Page* 289, doth ſuppoſe the Wine Gallon to contain but 224, or 225 Cubic Inches at the moſt, and purſuant to this Account an Experiment was made by Mr. *Richard Walker* and Mr. *Philip Shales*, two General Officers in the Exciſe. They cauſed a Veſſel to be very exactly made of Braſs, in Form of a Parallelopipedon, each Side of its Baſe was 4 Inches, and its depth 14 Inches; ſo that its juſt Content was 224 Cubic Inches. This Veſſel was produced at *Guild Hall* in *London* (*May* 25th, 1688) before the *Lord-Mayor*, the *Commiſſioners* of *Exciſe*, the Reverend Mr. *Flamſteed* Aſtr. Reg. Mr. *Halley*, and ſeveral other ingenious Gentlemen

Gentlemen, in whose Presence Mr. *Shales* did exactly fill the aforesaid Brazen Vessel with clear Water, and very carefully emptied it into the old Standard Wine Gallon kept in *Guild-Hall*, which did so exactly fill it, that all then present were fully satisfied the Wine Gallon doth contain but 224 Cubic Inches. (*This notable Experiment I saw tried.*) However, for several Reasons, it was at that Time thought convenient to continue the Former supposed Content of 231 Cubic Inches to be the Wine Gallon, and that all Computations in Gaging should be made from thence, as above.

The Beer or Ale Gallon (which are both one) is much larger than the Wine Gallon, it being (as I presume) made at first to correspond with *Averdupois Weight*, as the Wine Gallon did with *Troy Weight*: For (as I said before *Page* 33) one Pound *Averdupois* is equal to 14 Ounces 12 Penny Weight *Troy*, very near.

And, as one Pound *Troy* is in Proportion to the Cubic Inches in a Wine Gallon, so is one Pound *Averdupois* to the Cubic Inches in an Ale Gallon. That is, $12 : 231 :: 14\frac{12}{20} : 281\frac{1}{2}$, very near the Cubic Inches contained in an Ale Gallon, as appears from an Experiment made by one *Nicholas Gunton*, General Gager in the Excise, about 60 Years ago, who by such a Vessel mentioned before in the last Page, did find the Standard Ale-Quart (kept in the Exchequer, *Vid.* 12. *Car.* 2) to contain just $70\frac{1}{2}$ Cubic Inches, consequently the Ale Gallon must contain 282 Cubic Inches, and from thence the following Tables are computed.

Ale-Measure.

Cubic Inches

282 = 1 *Gallon*
2256 = 8 = 1 *Ferkin*
4512 = 16 = 2 = 1 *Kilderkin*
9024 = 32 = 4 = 2 = 1 *Barrel*
13536 = 48 = 6 = 3 = 1½ = 1 *Hogshead.*

Note, A Ferkin of Soap and of Herrings are the same with that of Ale.

Beer-Measure.

Cub. Inches

282 = 1 *Gallon*
2538 = 9 = 1 *Ferkin*
5076 = 18 = 9 = 1 *Kilderkin*
10152 = 36 = 4 = 2 = 1 *Barrel*
15228 = 54 = 6 = 3 = 1½ = 1 *Hogshead.*

 N. B.

N. B. This Diſtinction or Difference betwixt *Ale* and *Beer-Meaſure*, is now only uſed in *London*. But in all other Places of *England* the following Table of *Beer* or *Ale*, whether it be ſtrong or ſmall, is to be obſerved, according to a Statute of Exciſe, made in the Year 1689.

Cub. Inches					
282	= 1 *Gallon*				
2397	= $8\frac{1}{2}$	= 1 *Ferkin*			
4794	= 17	= 2	= 1 *Kild.*		
9588	= 34	= 4	= 2	= 1 *Barrel*	
14382	= 51	= 6	= 3	= $1\frac{1}{2}$	= 1 *Hogſhead.*

7. Of *Dry Meaſure*.

Dry Meaſure is different both from the *Wine* and *Ale-Meaſure*, being as it were a Mean betwixt both, tho' not exactly ſo; which upon Examination I find to be in Proportion to the aforeſaid old Standard Wine Gallon, as *Averdupois Weight* is to *Troy Weight*; That is, as one Pound *Troy* is to one Pound *Averdupois*, ſo are the Cubic Inches contained in the old Wine Gallon, to the Cubic Inches contained in the Dry or Corn Gallon.

Viz. $12 : 14\frac{12}{20} :: 224 : 272\frac{1}{2}$ which is very near to $272\frac{1}{4}$, the common received Content of a Corn Gallon, altho' now it is otherwiſe ſettled by an Act of Parliament made in *April* 1697, the Words of that Act are theſe:

Every round Buſhel *with a plain and even Bottom, being made eighteen* Inches *and a half wide throughout, and eight* Inches *deep, ſhould be eſteem'd a legal* Wincheſter Buſhel, *according to the* Standard *in his* Majeſty's Exchequer.

Now a Veſſel being thus made will contain 2150,42 Cubic Inches, conſequently the Corn Gallon doth contain but $268\frac{4}{5}$ Cubical Inches.

Cub. Inches				
268,8	= 1 *Gallon*			
537,6	= 2	= 1 *Peck*		
2150,4	= 8	= 4	= 1 *Buſhel*	
17203,2	= 64	= 32	= 8	= 1 *Quarter.*

Note, { 4 Buſhels = a Comb.
10 Combs = a Wey, and
2 Weys = a Laſt of Corn. }

I obſerv'd amongſt the Lead-Mines in *Derbyſhire*, (*Anno* 1692) that the Miners bought and ſold their Lead-Ore, by a Meaſure which they call'd an Ore Diſh; whoſe Dimenſions I carefully took and found them

Thus { Length 21.3 / Breadth 6. / Depth 8.4 } *Inches*

Conſe-

Consequently it's Content is 1073,52 Cubic Inches, which is very near equal to 4 Corn Gallons, according to the abovementioned Settlement.

Nine of those Dishes they call a Load of *Ore*, which if it be pretty good, will produce about 3 hundred Weight of Lead.

8. *Of* Time.

It is not an easy Thing to give a true Definition of *Time*; for (according to the *Philosophic Poet*)

' Time *of it self is nothing, but from Thought*
' *Receives its Rise, by labouring Fancy wrought*
' *From Things consider'd, whilst we think on some*
' *As present, some as past, or yet to come.*
' *No Thought can think on* Time, *that's still confest,*
' *But thinks on Things in Motion or at Rest.*

And so on, *Vide Lucretius*, Book I.

That is, Time only shews the Duration or Mutation of Things, a Year being the Standard or Integer, by which such Continuance or Change is computed. And a Year is that Space of Time in which the Sun (apparently) completes its Revolution from any one Point in the Ecliptic (an imaginary Circle in the Heavens) to the same Point again, which according to Modern Observations is perform'd in 365 Days, 5 Hours, 48 Minutes, 57 Seconds, 21 Thirds, *&c.* But a Second being the least Part of Time that can be truly measured by the Motion of any Mechanical Engin, as a Clock, *&c.* (a Third being less than the Twinkling of an Eye) I begin the following Table with Seconds.

Seconds. ''				
60 =	1 ' *Minutes*			
3600 =	60 =	1 ° *Hours*		
86400 =	1440 =	24 =	1 *Days* ° ' ''	
31556937 =	525349 =	8765 =	365 + 5 + 48 + 57 =	1 *Year*, *call'd a Solar Year.*

But the common Year, usually call'd the *Julian* Year, doth consist of 365 Days and 6 Hours, and is divided into twelve unequal Months, called Calendar Months, whose Names and Number of Days are the Subject of every *Almanack*.

To

To these Tables it may not be amiss to give a brief Account of such Coins, Weights and Measures as are frequently mention'd in the Scriptures. As I have deduc'd them from those which seem to be the most correct, inserted in the *Index* to the large *Bible*, Printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough*.

The *Hebrew* Weights, compared with { *Troy Weight.*

	Oz.	*Pw.*	*Grains.*
A Gerah=	0	. 0	. $10\frac{1}{2}\frac{2}{3}$
10 Gerahs=a Bekah=	0	. 4	. $13\frac{1}{2}$
2 Bekahs=a Shekel=	0	. 9	. 3
100 Shekels=a Menah=	45	. 12	. 12

Note, A *Shekel* is said to be their Original Weight.

Their Coin. { *English Coin.*

	l.	*s.*	*d.*	
A Silver Menah=	7	. 1	. $5\frac{1}{4}$	Weight 60 Shekels.
Talent of Silver=	357	. 11	. $10\frac{1}{2}$	Weight is 3000 Shekels.
Talent of Gold=	5075	. 15	. $7\frac{1}{2}$	The same Weight men-
The Gold Dram=	1	. 0	. 4	tioned in *Ez.* 2. 19.

The *Roman* Money mentioned in the *New Testament*.

A Denarius, or Silver Penny=7 *d.* 3 Farthings.
Asses of Copper=0 . 3 Farthings.
Assarium=0 . 1 $\frac{1}{2}$ Farthing.
Quadrans=0 . $\frac{3}{4}$ of a Farthing.
A Mite=0 . $\frac{1}{3}$ of a Farthing.

Their *Long Measures* compared with { *English Measure.*

	Yar.	*Feet.*	*Inch Parts*
A Finger's Breadth=	0.	0 .	0,912
4 Fingers=a Hand's Breadth=	0.	0 .	3,648
2 Hands=the least Span=	0.	0 .	7,296
3 Hands Breadth=the longest Span=	0.	0 .	10.944
2 Spans=the longest Cubit=	0.	1 .	9 888
4 Cubits=a Fathom=	2.	1 .	3,552
6 Cubits=*Ezekiel*'s Reed=	3.	1 .	11 328
400 Cubits=a Stadium=	243.	0 .	7.2
10 Stadiums=a Mile=	2432.	0 .	0
3 Miles=a Parasang=	7296.	0 .	0.
Which is 4 *English* Miles and	256.		

Their

Their Measures of Capacity, compared with English Wine.	Gal.	Pints	Inch.
A Cotyla =	0	$0\frac{1}{2}$	3.037
A Log =	0	$0\frac{1}{2}$	9,83
4 Logs = a Cab =	0	3	10,458
10 Cotyla's = an Omer =	0	6	1,5
3 Cabs = a Hin =	1	2	2,5
2 Hins = a Seah =	2	4	5,
3 Seahs = an Epha =	7	4	15,
10 Epha's = a Chomer =	75	5	5,625

SECT. 2. Addition of *Weights*, &c.

The foregoing Tables being ſo well underſtood, as that you can readily tell (without pauſing) how many Units of any one Denomination, do make one of the next ſuperior Denomination (eſpecially in thoſe Tables as are moſt uſeful for your Buſineſs) it will then be as eaſy to add, or ſubtract them, as to add, or ſubtract whole Numbers, due care being taken in placing all Numbers that are of one Denomination exactly underneath each other. That is to ſay, in Money place Pounds under Pounds, Shillings under Shillings, Pence under Pence, *&c.* Underſtand the like in Weights and Meaſures, *&c.* According to their ſeveral Denominations: Then in Addition obſerve this *Rule.*

RULE.

Always begin with thoſe Figures of the loweſt or leaſt Denomination, and add them all together into one Sum, then conſider how many of the next ſuperior Denomination are contained in that Sum, ſo many Units you muſt carry to the ſaid next ſuperior Denomination to be added together with thoſe Figures that ſtand there; and if any thing remain over or above thoſe Units ſo carried, that Overplus muſt be ſet down underneath its own Denomination: And ſo proceed on from one Denomination to another until all be finiſhed.

Example in Coin.

Let it be required to add 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17 *s.* 09 *d.* into one Sum.

Theſe particular Sums being placed, as before directed, will ſtand as in the Margin following.

Then according to the Rule, I begin with the Pence (being here the loweſt or leaſt Denomination) and adding them all together, I find their Sum to be 29 *d.* that is 2 *s.* and 5 Pence; (for

(for 24 *d.* = 2 *s.* and 29 — 24 = 5) the 5 *d.* I set down underneath its own Denomination, and carry the 2 *s.* to the Place of Shillings, adding them and all the Shillings together, I find the Sum to be 48 *s.* *viz.* 2 *l.* 8 *s.* I set down the 8 *s.* underneath its own Place of Shillings, and carry the 2 *l.* to the Place of Pounds, adding them and all the Pounds together, I find their Sum is 128 *l.* consequently the total Sum required is 128 *l.* 08 *s.* 05 *d.*

l.	*s.*	*d.*
35	14	06
27	02	10
54	13	04
10	17	09
128	08	05

Now, for as much as it often happens in keeping Books of Accompts, (and in other Business) that it is required to add up large Sums of Money, consisting of 30, 40, or more several particular Sums, nay, perhaps filling up the whole length of a Sheet of Paper, I humbly conceive in those Cases the best and easiest Way will be to part them into Parcels, not exceeding 10 or 12 particular Sums in each Parcel, and Sum up (on a by-Paper) the Particulars in each Parcel; that done, add together all the Sums of those Parcels into one Sum, and that will be the total Sum required.

Also to avoid the making of Points, or other Marks amongst your Figures, it will be convenient to get the following Tables by heart.

The Pence-Table.

d.	*s.*	*d.*	*s.*
12 =	1.	72 =	6.
24 =	2.	84 =	7.
36 =	3.	96 =	8.
48 =	4.	108 =	9.
60 =	5.	120 =	10.

The Shillings Table.

s.	*l.*	*s.*	*l.*
20 =	1.	120 =	6.
40 =	2.	140 =	7.
60 =	3.	160 =	8.
80 =	4.	180 =	9.
100 =	5.	200 =	10.

The Use of these Tables is so obvious, that I presume 'tis needless to explain them.

Examples in Addition *of* Weights.

Troy Weight.

℔	*Oz.*	*Pw.*	*Gr.*
3	09	00	10
5	08	15	21
10	10	12	22
0	11	19	23
Sum 21	04	09	04

Averdupois Weight.

Tun.	*C.*	*Q.*	℔	*Oz.*
12	15	2	24	12
7	10	3	21	15
0	18	1	14	11
1	19	3	27	15
Sum 23	05	3	05	05

Example

Examples in Addition of long Measure.

Yards	*Qrs.*	*Nails*		*Miles.*	*Fur.*	*Poles*	*Yards*	*Feet*	*Inch.*
35	2	3		2	6	32	4	2	9
17	3	1		0	7	27	3	1	10
129	1	2		1	3	39	1	2	11
182	3	2	Sum	5	2	19	5	0	0

I think it needlefs to fet down more Examples of this kind, for if thefe 5, efpecially the laft, be well underftood, they will be fufficient to fhew how any other may be performed.

Sect. 3. Subtraction *of Weights*, &c.

Subtraction is but the Converfe of the precedent Work, and may be performed by obferving this *Rule.*

RULE.

Begin with the loweft or leaft Denomination, as before in Addition, and take or fubtract the Figure, or Figures, in that Place of the Subtrahend, from the Figure, or Figures, that ftand over them of the fame Denomination; fetting down the Remainder (as in *Page* 12.) But if that cannot be done, then you muft increafe the upper Figure, or Figures, with one of the next fuperior Denomination, and from that Sum make Subtraction; and fo proceed to the next fuperior Denomination, where you muft pay the one borrowed, by adding Unity to the Subtrahend in that Place, *&c.* as in whole Nnmbers.

Example in Coin.

	l.	*s.*	*d.*		*l.*	*s.*	*d.*
From	386	09	08	From	569	10	06
Take	173	04	06	Subt.	389	15	08
Remains	213	05	02		179	14	10

The firft of thefe Examples is felf-evident. In the fecond Example, beginning at the Place of Pence, being here the leaft Denomination, I am to take 8 *d.* from 6 *d.* but becaufe that cannot be done, I muft, according to the Rule, borrow one of the next Denomination, *viz.* 1 *s.* and add it to the 6 *d.* which makes it 18 *d.* (for 1 *s.*=12 *d.* and 12+6 *d.*=18 *d.*) then I take 8 *d.* from that 18 *d.* and there remains 10 *d.* to be fet down underneath the Place of Pence; that done, I proceed to the Place of Shillings, where I muft now pay the 1 *s.* faying one borrowed, and 15 makes 16 from 10 cannot be, but 16 from 30 and there remains 14. That is, I

 borrow

borrow one of the next Denomination, *viz.* 1 *l.* and add to it the 10 *s.* which makes it 30 *s.* (for 1 *l.*=20 *s.* and 20+10=30) having set down the remaining 14 *s.* underneath its own Place of Shillings, I proceed to the Place of Pounds, where paying the 1 *l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there remains 9, and so on as in whole Numbers, until all be finished; and the Remainder will be 179 *l.* 14 *s.* 10 *d.*

This Example being a little considered will render all others in this Rule easie.

Examples in Weight.

Troy Weight.

	℔	*oz.*	*pwt.*	*gr.*
From	9	10	16	18
Take	5	09	18	22
	4	00	17	20

Averdupois Weight.

c.	*qr.*	℔	*oz.*
17	2	15	10
14	3	18	12
2	2	24	14

Examples in long Measure.

	yards	*qrs.*	*nails*
From	78	3	2
Take	29	3	3
Rests	48	3	3

miles	*fur.*	*pol.*	*yds.*	*feet*	*inches*
22	3	26	3	1	11
18	6	29	4	2	7
3	4	36	4	0	10

Example in Time.

	days	°	′	″
From	27	18	35	21
Subtract	16	21	46	36
Remains	10	20	48	45

The Proof of Addition and Subtraction in these Numbers of different Denominations is the very same with that of whole Numbers in *Page* 13. I shall therefore refer you to that Place, and omit repeating it here.

Sect 4. *Of* Reduction.

By Reduction, Numbers of different Denominations are brought into one Denomination.

That is, it alters or changes any superior Denomination proposed, into any inferior or lesser Denomination required; still keeping

keeping them equivalent in Value. And by that means they become fitly prepared for Multiplication and Divifion; which otherwife could not fo conveniently be performed. Therefore the Bufinefs of Reduction is very ufeful in the Rule of Proportion, commonly called the *Golden Rule*, or *Rule of Three*, efpecially to thofe who do not underftand either Vulgar or Decimal Fractions: And 'tis thus performed.

RULE.

Confider how many Units of the Denomination required, make one of that Denomination propofed to be reduced; which is eafily known by its refpective Table, and with that Number of Units, multiply the Denomination propofed, and their Product will be the Number required.

Example in Coin.

Let it be required to reduce or change 357 *l.* into Shillings, and thofe Shillings into Pence, which fhall ftill be equal in Value with the 357 *l.*

	357	
Multiply with	20	the Shillings in one Pound
	7140	= the Shillings in 357 *l.*
Multiply with	12	the Pence in one Shilling
	1428	
	714	
	85680	= the Pence in 357 *l.* as was required.

Or 357 *l.* may be reduced into Pence, at one Operation: Thus,

	357 *l.*	
Multiply with	240	the Pence contained in one Pound.
	1428	
	714	
	85680	= the Pence in 357 *l.* as before.

But when the Numbers propofed to be reduc'd are of feveral Denominations, and it is required to bring them all to the loweft; you muft reduce the higheft or greateft Denomination to the next lefs, Adding the Numbers that are of that lefs Denomination together, then reduce their Sum to the next lower Denomination, Adding together all the Numbers that are of that Denomination, and fo proceed gradually on until all is done.

Example.

Let it be required to reduce 375 *l.* 17 *s.* 10 *d.* 3 *q.* into one Denomination, *viz.* into Farthings.

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20

7500 = the Shillings in 375 *l.*
+ 17 *s.*

7517 = the Shillings in 375 *l.* 17 *s.*
12

15034
7517

90204 = the Pence in 375 *l.* 17 *s.*
+ 10 *d.*

90214 = the Pence in 375 *l.* 17 *s.* 10 *d.*
4

360856 = the Farthings in 375 *l.* 17 *s.* 10 *d.*
+ 3 *q.*

360859 Farthings = 375 *l.* 17 *s.* 10 *d.* 3 *q.* as was required.

The Work of this Example, and all other Operations of this kind, may be somewhat shortned by observing the following Method.

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20 Multiply and add in the 17 *s.*

7517
12 Multiply and add in the 10 *d.*

15034
7518

90214
4 Multiply and add in the 3 *qrs.*

360859 the Farthings as before.

Example in Troy Weight.

Suppose it be required to reduce 29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.* into the least Denomination, *viz.* into Grains,

Thus,

Thus,	29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.*
Multiply with	12 the *oz.* in 1 ℔ and add in the 8 *oz.*
	66
	29
	356=the Ounces in 29 ℔ 8 *oz.*
Multiply with	20 the *pwts.* in 1 *oz.* and add in the 18 *pwt.*
	7138=the *pwts.* in 29 ℔ 8 *oz.* 18 *pwt.*
Multiply with	24 the Grains in 1 *pwt.* and add in the 21 *gr.*
	28553
	14278
	171333 the Grains=29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.*

Theſe two Examples at large being well underſtood, may ſuffice to ſhew how all Operations of this kind are performed; either in Weights, Meaſures, or Time. I ſhall only inſert a few Examples of each Sort for the Learner's Practice.

1. In 23 *C.* 3 *qrs.* 21 ℔ 9 *oz.* Averdupois Weight; How many Ounces? *Anſwer.* 42905 Ounces.

2. In 252 *Engliſh* Miles, How many Yards, Feet and Inches? *Anſwer* 443520 Yards=1330560 Feet=15966720 Inches.

3. In 1692 common Years, How many Days, Hours, and Minutes? *Anſwer* 618003 Days, 14832072 Hours, 889924320 Minutes.

Note, a common Year=365 Days, 6 Hours, ſee *Page* 37.

4. In 5786 Pounds, 17 Shillings, 9 Pence Sterling: How many Shillings, Pence and Farthings?

Anſwer, 115737 Shillings, 1388853 Pence, or 5555412 Farthings. That is, 5786 *l.* 17 *s.* 9 *d.*=115737 *s.* 9 *d.*=1388853 *d.* &c.

The next thing will be to ſhew how to bring Numbers from a leſſer to a greater Denomination, which by moſt Authors is called, tho' very improperly.

Reduction *Aſcending.*

This Work is the Converſe of the laſt, and is performed by Diviſion. Thus,

RULE.

Conſider how many of the Denomination propoſed make one of the Denomination required, and make that Number your Diviſor, by which divide the Denomination propoſed; and the Quotient will be the Number required.

Ex-

Example.

Let it be required to find how many Shillings and Pounds are contained in 85680 Pence.

The Pence in 1 *s.* are 12) 85680 (7140 *s.*=85680 *d.*

Again the Shillings in 1 *l.* are 20) 7140 (357 *l.* the Anſwer required.

Another Example in Coin.

How many Pence, Shillings and Pounds, are contained in 264859 Farthings.

```
             12)           20)
4) 264859 ( 66214  d. ( 5517 s. ( 275 l.
    24        62         151
    08        21         117
     05        94        (17) s.
      19     (10) d.
Remains (3) q.
```

Note, The Remainder is always of the ſame Denomination with the Dividend.

The laſt Quotient 275 *l.* together with the ſeveral Remainders, gives the Anſwer required.

Viz. 275 *l.* 17 *s.* 10 *d.* 3 *q.*=264859 Farthings.

Example in Troy Weight.

Suppoſe it were required to find how many *pwts. ozs.* and ℔s. are contained in 171333 Grains.

```
             20)               12)
24) 171333 gr. (7138 pw.  (356 (29 ℔
    168...       113        24
     33          138        116
     24                     108
               (18) pwt.
      93                   (8) oz.
      72
      213
      192
Remains (21) gr.
```

Anſwer, 29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.* This and the laſt Example are the Reverſe or Proof of thoſe in *Page* 43, 45.

1. In 42905 Ounces Averdupois Weight; How many Pounds, &c.

Thus

```
                28)           4)
Thus  16) 42905   (2681 lb      (95 qrs.  (23 C.
          -----                 ------
           109     252           15
          -----   -----         ------
           130     161          (3)
          -----    140
            25    -----
          -----
           (9)    (21) Answer, 23 C. 3 qrs. 21 lb 9 oz.
```

2. In 15966720 Inches; How many *English* Miles, *&c.*
Answer, 252 Miles, *&c.* As Occasion requires.

There are many useful Questions may be answered by Help of Reduction only: As the Changing of one Sort of Coin for another; and comparing of one Sort of Measure with another, *&c.*

For Instance; Suppose one had 347 Rix-Dollars, at 4 *s.* 6 *d. per* Dollar: and desired to know how many Pounds *Sterling* they make.

```
      347
       54=the Pence in one Dollar, viz. 4 s. 6 d.=54 d.
     ----
     1388
     1735    20)
12) 18738 d. (1561 s.  (78 l.
    ----     ------
      67       161
    ----     ------
      73       (1)s.
    ----
      18
    ----
     (6)d.
```

Answer, 78 *l.* 1 *s.* 6 *d. Sterling*, are= 347 Rix-Dollars.

Quest. 2. In 645 *Flemish* Ells; How many Ells *English*?

Note, 3 Quarters of a Yard *English* make one Ell *Flemish*, and 1¼ or 5 Quarters of a Yard is an *English* Ell.

```
        Therefore, 645
                     3=the qrs. of a Yard in 1 Ell Flemish.
qrs. in 1 Ell=5) 1935 (387 English Ells for the Answer.
```

Quest. 3. Suppose a Bill of Exchange were accepted at *London*, for the Payment of 400 *l. Sterling*, for the Value deliver'd at *Amsterdam* in *Flemish* Money at 1 *l.* 13 *s.* 6 *d.* for 1 *l. Sterling*. How much *Flemish* Money was delivered at *Amsterdam*?

First, 1 *l.* 13 *s.* 6 *d.*=402 *d.* the Value of one Pound *Sterling* at *Amsterdam*.

Then 402 *d.*×400=160800 *d.*=670 *l. Flemish*, and so much was deliver'd at *Amsterdam*.

CHAP.

CHAP IV.

Of Vulgar Fractions.

Sect. 1. *Of* Notation.

A Fraction, or broken Number, is that which repreſents a Part or Parts of any thing propoſed (*vide Page* 3.) and is expreſſed by two Numbers placed one above the other with a Line drawn betwixt them.

Thus, $\left\{ \frac{3}{4} \right.$ *Numerator.* / *Denominator.*

The Denominator or Number placed underneath the Line, denotes how many equal Parts the Thing is ſuppoſed to be divided into (being only the Diviſor in Diviſion.) And the Numerator or Number placed above the Line, ſhews how many of thoſe Parts are contained in the Fraction, it being the Remainder after Diviſion. (See *Page* 29.) And theſe admit of three Diſtinctions:

Viz. { *Proper* or *Simple*, / *Improper*, / *Compound*, } *Fractions.*

A proper, pure, or ſimple Fraction, is that which is leſs than an Unit. That is, it repreſents the immediate Part or Parts of any thing leſs than the Whole, and therefore its Numerator is always leſs than the Denominator.

As { $\frac{1}{4}$ is one *Fourth Part.* / $\frac{1}{3}$ is one *Third Part.* } And { $\frac{1}{2}$ is one *Half.* / $\frac{2}{3}$ is Two *Thirds*, &c.

An improper Fraction is that which is greater than an Unit; That is, it repreſents ſome Number of Parts greater than the whole thing; and its Numerator is always greater than the Denominator.

As $\frac{5}{3}$ Or $\frac{9}{7}$ Or $\frac{41}{15}$ &c.

A Compound Fraction is a Part of a Part, conſiſting of ſeveral Numerators and Denominators connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, *&c.* and are thus read, The one Third of the three Fourths of the two Fifths of an Unit.

That is, when a Unit (or whole Thing) is firſt divided into any Number of equal Parts, and each of thoſe Parts are ſubdivided

ſubdivided into other Parts, and ſo on: Then thoſe laſt Parts are called compound Fractions, or Fractions of Fractions.

As for inſtance, ſuppoſe a Pound *Sterling*, or 20 s. be the Unit or Whole; then is 8 s. the $\frac{2}{5}$ of it, and 6 s. the $\frac{3}{4}$ of thoſe two Fifths, and 2 s. is the $\frac{1}{3}$ of thoſe three Fourths. *Viz.* 2 s. $=\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of one Pound *Sterling*.

All compound Fractions are reduced to ſingle ones, Thus,

RULE.

Multiply *all the* Numerators *into one another for a* Numerator, *and all the* Denominators *into one another for the* Denominator.

Thus the $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{60}$. Or $\frac{1}{10}$.
For $1\times3\times2=6$ the Numerator, and $3\times4\times5=60$ the Denominator, but $\frac{6}{60}$ or $\frac{1}{10}$ of a Pound *Sterling* is 2 s. As above,

---

Sect. 2. To Alter or Change different Fractions into one *Denomination* retaining the ſame Value.

In order to gain a clear Underſtanding of this Section, it will be convenient to premiſe this Propoſition, *viz.* "If a Number " multiplying two Numbers produce other Numbers, the Numbers " produced of them ſhall be in the ſame Proportion that the " Numbers multiplied are, 17 *Euclid* 7.

That is to ſay, If both the Numerator and Denominator of any Fraction be equally multiplied into any Number, their Products will retain the ſame Value with that Fraction.

As in theſe. $\frac{2\times2}{3\times2}=\frac{4}{6}$. Or $\frac{2\times3}{3\times3}=\frac{6}{9}$. Or $\frac{2\times5}{3\times5}=\frac{10}{15}$, &c,

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the ſame Value in reſpect to the Whole or Unit.

From hence it will be eaſy to conceive how two, or more Fractions that are of different Denominations, may be alter'd or chang'd into others that ſhall have one common Denominator, and ſtill retain the ſame Value.

Example. Let it be required to change $\frac{2}{3}$ and $\frac{3}{7}$ into two other Fractions that ſhall have one common Denominator, and yet retain the ſame Value.

According to the foregoing Propoſition, if $\frac{2}{3}$ be equally multiplied with 7, it will become $\frac{14}{21}$ *viz.* $\frac{2\times7}{3\times7}=\frac{14}{21}$. Again if $\frac{3}{7}$ be equally multiplied with 3, it will become $\frac{9}{21}$ *viz.* $\frac{3\times3}{7\times3}=\frac{9}{21}$. And by this means I have obtained two new Fractions $\frac{9}{21}$ and $\frac{14}{21}$ that are of one Denomination, and the ſame Value with the two firſt propoſed, *viz.* $\frac{14}{21}=\frac{2}{3}$ and $\frac{9}{21}=\frac{3}{7}$.

And from hence doth arise the General Rule for bringing all Fractions into one Denomination.

RULE.

Multiply all the Denominators *into each other for a new (and common)* Denominator. *And each* Numerator *into all the* Denominators *but its own, for New* Numerators.

Example. Let the proposed Fractions be, $\frac{1}{3}$. $\frac{2}{5}$. $\frac{3}{4}$. and $\frac{6}{7}$.

Then by the *Rule.*

A new Denominator will be thus found.

3
5
15
4
60
7
420

And the new Numerators will be thus found.

1.	2.	3.	6.
5	3	3	3
5	6	9	18
4	4	5	5
20	24	45	90
7	7	7	4
140.	168.	315.	360.

Hence 420 is the common Denominator. And 140. 168. 315. 360 are the new Numerators, which being placed Fraction-wise are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$. the new Fractions required.

That is, $\frac{140}{420}=\frac{1}{3}$. $\frac{168}{420}=\frac{2}{5}$. $\frac{315}{420}=\frac{3}{4}$. and $\frac{360}{420}=\frac{6}{7}$.

Sect. 3. *To bring mix'd* Numbers *into* Fractions, *and the contrary.*

Mix'd Numbers are brought into improper Fractions by the following

RULE.

Multiply *the* Integers *or whole* Numbers, *with the* Denominator *of the given* Fraction, *and to their Product add the* Numerator, *the Sum will be the* Numerator *of the* Fraction *required.*

Example. $9\frac{4}{5}$ by the Rule will become $\frac{49}{5}$. For $9\times5=\frac{45}{5}$.
And, $\frac{45}{5}+\frac{4}{5}=\frac{49}{5}$ the improper Fraction required.
Again, $13\frac{11}{15}$ will become $\frac{206}{15}$. For $13\times15=\frac{195}{15}$.
And $\frac{195}{15}+\frac{11}{15}=\frac{206}{15}$. And so for any other as Occasion requires.
To find the true Value of any improper Fraction given is only the Converse of this Rule. For if $\frac{49}{5}=9\frac{4}{5}$ as before is evident;

evident: Then it follows that if 49 be divided by 5, the Quotient will give $9\frac{4}{5}$. And if 206 be divided by 15 it will give $13\frac{11}{15}$, &c. consequently it follows, That

If the Numerator of any improper Fraction be divided by its Denominator, the Quotient will discover the true Value of that Fraction.

Examples.

$\frac{35}{7}=5$. And $\frac{50}{9}=5\frac{5}{9}$. And $\frac{121}{20}=6\frac{1}{20}$. Or $\frac{15}{4}=3\frac{3}{4}$, &c.

When whole Numbers are to be express'd Fraction-wise, it is but giving them an Unit for a Denominator. Thus 45 is $\frac{45}{1}$. 9 is $\frac{9}{1}$ and 25 is $\frac{25}{1}$, &c.

Sect. 4. *To* Abbreviate *or* Reduce Fractions *into their lowest or least* Denomination.

This is done, not out of any Necessity, but for the more convenient Managing of such Fractions as are either proposed in large Terms; or swell into such, either by Addition or otherwise: Besides, 'tis most like an Artist to express or set down all Fractions in the lowest Terms possible; And to perform that, it will be necessary to consider of these following Propositions.

Numbers are either Prime or Composed.

1. A PRIME Number is that which can only be measured by an Unit. *Euclid* 7. *Defi.* 11.

That is, 3. 5. 7. 11. 13. 17. &c. are said to be Prime Numbers, because it is not possible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are such as only an Unit doth measure, being their common Measure. *Euclid* 7. *Defi.* 12.

For instance, 7 and 13 are Prime Numbers to each other, because they cannot be divided by any Number but an Unit. And 9 and 14 are also Prime Numbers to each other, for altho' 3 will measure or divide 9 without leaving a Remainder, yet 3 will not measure 14 without leaving a Remainder: Again, altho' 2 will measure 14 without any Remainder, yet 2 will not measure 9 without leaving a Remainder, &c.

3. A COMPOSED Number is that which some certain Number measureth. *Euclid* 7. *Defi.* 13.

For instance, 15 is a composed Number of 3 and 5, for $5\times3=15$, consequently 3 or 5 will justly measure 15. Also 20

is composed of 5 and 4, *viz.* $5 \times 4 = 20$, therefore 5 and 4 will each justly measure 20.

4. Numbers composed the one to the other, are they which some Number being a common Measure to them both doth measure. *Euclid* 7. *Defi.* 14.

That is, If two or more Numbers can be divided by one and the same Divisor; then are those Numbers said to be composed one to another.

For instance, 14 and 21 are Numbers composed the one to the other, because they can both be measured or divided by 7. For $7 \times 2 = 14$, and $7 \times 3 = 21$, therefore 7 is a common Measure to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated, it will become $\frac{2}{3}$.

Thus $\left\{ \begin{array}{l} 7)\ \dfrac{14 = 2}{21 = 3} \\ 7) \end{array} \right.$

And how those greatest common Measures may be found, comes from *Euclid* 7. *pro.* 1. 2. 3. and is thus:

RULE.

Divide *the greater* Number *by the lesser, and that* Divisor *by the* Remainder (*if there be any*) *and so on continually until there be no* Remainder *left: Then will this last* Divisor *be the greatest Common* Measure (*and if it happen to be* 1, *then are those* Numbers *Prime* Numbers, *and are already in their lowest Terms, but if otherwise*) Divide *the* Numbers *by that last* Divisor, *and their* Quotient *will be their least Terms required.*

Example.

Let it be required to find the greatest common Measure of 72 and 108, *viz.* Of $\frac{72}{108}$.

```
72) 108 (1
    72
   ----
36) 72 (2
    72
   ----
    (0)
```

Here because there's no Remainder; 36 is the greatest common Measure.

Therefore, $\left\{ \begin{array}{l} 36)\ \dfrac{72 = 2}{108 = 3} \\ 36) \end{array} \right.$ Hence $\frac{72}{108}$ is abbreviated to $\frac{2}{3}$ the lowest Terms.

Again, to find the greatest common Measure of 744 and 899.

Thus,

```
Thus, 744) 899 (1
           744
           ---
           155) 744 (4
                620
                ---
                124) 155 (1
                     124
                     ---
                      31) 124 (4
                          124
                          ---
                          (0)
```

Here 31 is found to be the greatest common Measure by which 744 and 899 may be abbreviated to 24 and 29 their lowest Terms.

Thus, $\frac{31}{31}$) $\frac{744}{899}$ (= $\frac{24}{29}$, &c.

Note, If the proposed Numbers be even, they may be brought lower by a continued Halving of them, so long as they can be halved, *viz.* divided by 2.

Example.

'Tis required to reduce $\frac{56}{84}$ to its least Terms.

First, $\frac{2}{2}$) $\frac{56}{84}$ (= $\frac{28}{42}$. Again $\frac{2}{2}$) $\frac{28}{42}$ (= $\frac{14}{21}$.

This done, you may easily perceive that 7 will be the common Measure to 14 and 21, *viz.* $\frac{7}{7}$) $\frac{14}{21}$ (= $\frac{2}{3}$, &c,

If the Numbers proposed to be reduced have each a Cypher or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{150}{300}$ will be $\frac{15}{30}$, And $\frac{200}{300}$ will be $\frac{2}{3}$, &c.

That is, $\frac{150}{300}=\frac{15}{30}=\frac{1}{2}$. and $\frac{200}{300}=\frac{2}{3}$. also $\frac{360}{400}=\frac{36}{40}=\frac{18}{20}=\frac{9}{10}$.

Sect. 5. Addition *of* Fractions.

What hath been done by the Rules in this Chapter; is chiefly to prepare and fit Fractions of different Denominations for Addition or Subtraction, as Occasion requires, *viz.* If they are compound Fractions, they must be reduced to simple or pure Fractions, *per Rule, Sect.* 1.

If they are of different Denominations, they must be altered or chang,d, *per Rule, Sect.* 2.

That is, all Fractions must be brought into one Denomination before they can either be added, or subtracted, and that being done, Addition is thus performed.

RULE.

Add *together all the* Numerators, *and their* Sum *will be a* New Numerator, *under which* subscribe *the* common Denomination.

Ex-

Examples in Simple Fractions.

Let it be propoſed to add $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{3}{4}$ together. Firſt $\frac{1}{3}=\frac{20}{60}$. $\frac{2}{5}=\frac{24}{60}$, and $\frac{3}{4}=\frac{45}{60}$. *per Sect.* 2.

Then $\frac{20}{60}+\frac{24}{60}+\frac{45}{60}=\frac{89}{60}$. the Sum required, which according to *Section* 3. is $1\frac{29}{60}$. *viz.* $\frac{89}{60}=1\frac{29}{60}$.

Examples in Compound Fractions.

Let it be required to add $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{3}{4}$ into one Sum. Firſt $\frac{2}{3}$ of $\frac{3}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ *per Sect.* 1. And (*per Sect.* 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$ *viz.* $\frac{3}{7}=\frac{6}{14}$ and $\frac{1}{2}=\frac{7}{14}$ but $\frac{7}{14}+\frac{6}{14}=\frac{13}{14}$ the Sum required, *viz.* $\frac{3}{7}+\frac{2}{3}$ of $\frac{3}{4}=\frac{13}{14}$.

Examples in Mix'd Numbers.

'Tis required to add $5\frac{2}{3}$ to $7\frac{3}{4}$. theſe *per Sect.* 3. will be $\frac{17}{3}$ and $\frac{31}{4}$ But $\frac{17}{3}$ and $\frac{31}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$. *per Sect.* 2.
Then $\frac{68}{12}+\frac{93}{12}=\frac{161}{12}$, and $\frac{161}{12}=13\frac{5}{12}$. the Sum required.

Or you may bring only the Fractions to one Denomination, Thus, $5\frac{2}{3}$ and $7\frac{3}{4}$ will become $5\frac{8}{12}$ and $7\frac{9}{12}$.

Then $5\frac{8}{12}+7\frac{9}{12}=12\frac{17}{12}$. That is, $13\frac{5}{12}$. As before.

Sect. 6. Subtraction *of* Fractions.

Rule { Subtract *one* Numerator *from the other* (*according as the Queſtion requires*) *and their Difference will be a new* Numerator, *under which* ſubſcribe *the common* Denominator *as in* Addition.

Example 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$. Firſt $\frac{2}{9}$ and $\frac{3}{7}$. *per Sect.* 2. will become $\frac{14}{63}$ and $\frac{27}{63}$. then $\frac{27}{63}-\frac{14}{63}=\frac{13}{63}$, that is, $\frac{3}{7}-\frac{2}{9}=\frac{13}{63}$. As was required.

Example 2.

'Tis required to ſubtract $\frac{2}{3}$ of $\frac{8}{9}$ from $\frac{13}{14}$. Firſt $\frac{2}{3}$ of $\frac{8}{9}=\frac{16}{27}$. *per Sect.* 1. Again $\frac{16}{27}$ and $\frac{13}{14}$ will become $\frac{224}{378}$ and $\frac{351}{378}$ *per Sect.* 2. Then $\frac{351}{378}-\frac{224}{378}=\frac{127}{378}$.

Example 3.

From $6\frac{1}{8}$ ſubtract $3\frac{10}{48}$. Firſt, $6\frac{1}{8}=\frac{49}{8}$. and $3\frac{10}{48}=\frac{163}{48}$. *per Rule Sect.* 3. Again, $\frac{49}{8}=\frac{2352}{384}$, and $\frac{163}{48}=\frac{1304}{384}$. *per Rule Sect.* 2. Then $\frac{2352}{384}-\frac{1304}{384}=\frac{1048}{384}=2\frac{280}{384}=2\frac{35}{48}$. Or otherwiſe thus: Firſt, $6\frac{1}{8}=5\frac{9}{8}$, then

then bring $\frac{9}{8}$ and $\frac{19}{48}$ into one Denomination, *viz.* $5\frac{9}{8}=5\frac{432}{384}$ and $3\frac{19}{48}=3\frac{152}{384}$.

Then $5\frac{432}{384}-3\frac{152}{384}=2\frac{280}{384}=2\frac{35}{48}$. As before.

Example 4.

Let it be required to ſubtract $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$ from 7.
Firſt, $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}=\frac{30}{189}$. And $7=6\frac{189}{189}$.
Then $6\frac{189}{189}-\frac{30}{189}=6\frac{159}{189}=6\frac{53}{63}=7-\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$. As was required.

If theſe few Examples be well underſtood, the whole Buſineſs of adding, and ſubtracting Vulgar Fractions will be eaſy; which is really much more difficult to perform than either Multiplication or Diviſion, as will appear in the next *Section.*

Sect. 7. Of Multiplication *of* Fractions.

In order to perform either Multiplication or Diviſion, you muſt prepare the Terms to be multiplied (or divided) thus ; Reduce compound Fractions to ſimple ones, *per Sect.* 1. Bring Mixt Numbers into Improper Fractions, and expreſs Whole Numbers Fractionwiſe, *per Sect.* 3. Alſo it will be convenient to Abbreviate them to their ſmalleſt Terms when it can be done. Then Multiplication may be thus performed.

Rule. Multiply *the* Numerators *one into another for a* New Numerator; *and the* Denominators *one into another, for a* New Denominator. *As in theſe*

Examples.

1. The Product of $\frac{2}{5}$ into $\frac{3}{7}=\frac{6}{35}$. That is, $\frac{2\times3}{5\times7}=\frac{6}{35}$.
2. And the Product of $\frac{9}{16}$ into $\frac{20}{27}=\frac{180}{432}$. Or $\frac{5}{12}$.
3. Again, the Product of $\frac{7}{11}$ into $\frac{2}{5}$ of $\frac{5}{7}=\frac{70}{385}$. Or $\frac{2}{11}$.
For $\frac{2}{5}$ of $\frac{5}{7}=\frac{10}{35}$. Then $\frac{7}{11}\times\frac{10}{35}=\frac{70}{385}=\frac{2}{11}$.
4. Let it be required to multiply 6 with $3\frac{2}{5}$. Theſe prepared for the Work will ſtand thus. $\frac{6}{1}\times\frac{17}{5}$.
viz. $6=\frac{6}{1}$ and $3\frac{2}{5}=\frac{17}{5}$. Then $\frac{6}{1}\times\frac{17}{5}=\frac{102}{5}$. or $20\frac{2}{5}$.
Or otherwiſe thus, $6\times3=18$: And $\frac{2}{5}\times6=\frac{12}{5}=2\frac{2}{5}$.
Then $18+2\frac{2}{5}=20\frac{2}{5}$. As before.
5. Let it be required to multiply $7\frac{4}{9}$ with $5\frac{3}{7}$.
Firſt $7\frac{4}{9}=\frac{67}{9}$. and $5\frac{3}{7}=\frac{38}{7}$. Then $\frac{67}{9}\times\frac{38}{7}=\frac{2546}{63}=40\frac{26}{63}$.

Now the Reaſon of this Rule for multiplying of Fractions, and conſequently of theſe Operations, and all other performed by it; will be evident from this following. *Viz.*

Viz. If $\frac{4}{2}$ be multiplied with $\frac{12}{3}$ according to the Rule, their Product will be $\frac{48}{6}$. But $\frac{48}{6}=8$.

Now $\frac{4}{2}=2$. and $\frac{12}{3}=4$ *per Sect.* 3. But $4\times2=8$. *Ergo*, &c.

Sect. 8. Division of Fractions.

The FRACTIONS being first prepared as before directed, Division may be thus performed:

Rule. { Multiply *the Numerator of the* Dividend *into the* Denominator *of the* Dividing Fraction *for a New* Numerator: *And* Multiply *the other* Numerator *and* Denominator *together for a New* Denominator.

Examples.

1. Let $\frac{6}{35}$ be divided by $\frac{3}{7}$ *viz.* $\frac{3}{7})\frac{6}{35}(\frac{42}{105}=\frac{2}{5}$ the Quotient.

That is, according to the Rule $6\times7=42$ the new Numerator, and $35\times3=105$, the new Denominator, *&c.* as above.

2. Let it be requir'd to divide $\frac{20}{27}$ by $\frac{5}{12}$. *viz.* $\frac{5}{12})\frac{20}{27}(\frac{240}{135}=1\frac{7}{9}$.

For $12\times20=240$ the new Numerator, and $27\times5=135$ the new Denominator, *&c.* as before.

3. Suppose it were required to divide $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.

First, $\frac{2}{5}$ of $\frac{5}{7}=\frac{10}{35}$. Then $\frac{10}{35})\frac{2}{11}(\frac{70}{110}=\frac{7}{11}$.

4. Let $20\frac{2}{5}$ be divided by $3\frac{2}{5}$ *viz.* $\frac{102}{5}$ by $\frac{17}{5}$.

For $20\frac{2}{5}=\frac{102}{5}$, and $3\frac{2}{5}=\frac{17}{5}$, Then $\frac{17}{5})\frac{102}{5}(=6$ the Quotient.

5 Let it be required to divide $40\frac{26}{63}$ by $5\frac{3}{7}$.

First, $40\frac{26}{63}=\frac{2546}{63}$, and $5\frac{3}{7}=\frac{38}{7}$. Then $\frac{38}{7})\frac{2546}{63}(\frac{17822}{2394}$.

But $\frac{17822}{2394}=7\frac{4}{9}$ the true Quotient required.

6. Suppose it were required to divide 13 by $\frac{5}{7}$.

First, $13=\frac{13}{1}$. Then $\frac{5}{7})\frac{13}{1}(\frac{91}{5}=18\frac{1}{5}$, the Quotient.

7. Again, let it be required to divide $\frac{5}{7}$ by 6.

Viz. $\frac{6}{1})\frac{5}{7}(\frac{5}{42}$ for the Quotient required.

N.B. From hence you may observe, that when any Whole Number is divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number propos'd to be divided: But if any Fraction be divided by a Whole Number, greater than 1. Then the Quotient will be less than the Dividend; As in the two last Examples.

As

As to the Reaſon (or Proof) of this Rule for dividing Fractions: 'Tis only the Converſe to that of Multiplication, and will be very evident from this following.

Let $\frac{48}{6}$ be divided by $\frac{4}{2}$. Which according to the Rule is thus, $\frac{4}{2}$) $\frac{48}{6}$ ($\frac{96}{24}=4$. The true Quotient. Now $\frac{48}{6}=8$. And $\frac{4}{2}=2$. *per Sect.* 3. Conſequently $\frac{48}{6}$ divided by $\frac{4}{2}$ is but the ſame with 8 divided by 2 *viz.* 2) 8 (4. The Quotient as before.

I could have inſerted Geometrical Demonſtrations, for both the Rules of Multiplication and Diviſion of Fractions; but ſuppoſing the Learner purely unacquainted with thoſe kind of Demonſtrations, I thought theſe might be more intelligible to him, eſpecially in this Place.

CHAP. V.

Of Decimal Fractions.

W*HEN, or by whom this excellent Invention of* Decimal Arithmetick, *was firſt introduced is uncertain, but doubtleſs its Improvements, and the Perfection 'tis now in, is owing to later Years.*

Sect. 1. *Of* Notation.

In Decimal Fractions, the Integer or Whole Thing (whether it be Coin, Weight, Meaſure, or Time, *&c.*) is ſuppos'd to be divided into ten equal Parts; and every one of thoſe ten Parts are ſuppoſed to be ſubdivided into other ten equal Parts, *&c. ad infinitum.*

The Integer being thus divided (by Imagination) into 10, 100, 1000, 10000, *&c.* equal Parts, becomes the Denominator to the Decimal Fraction.

Thus, $\frac{2}{10}$. $\frac{3}{100}$. $\frac{7}{1000}$. $\frac{53}{10000}$. $\frac{745}{100000}$. *&c.*

Now theſe Denominators are ſeldom or never ſet down but only the Numerators; and thoſe are either diſtinguiſhed, or ſeparated from Whole Numbers by a Point or a Comma.

Thus, 5,4 is $5\frac{4}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is $35\frac{5}{100}$, *&c.*

But before we proceed further in Notation, it will be convenient for the Learner to conſider of the following Table (taken out of the learned Mr. *Oughtred's Clavis Mathematica*) which ſhews the very Foundation of Decimal Fractions.

Whole Numbers						*Decimal Parts*					
5	4	3	2	1	0,	1	2	3	4	5	6
Hundreds of Thousands. &c.	*Tens of Thousands.*	*Thousands.*	*Hundreds.*	*Tens.*	*Units Place.*	*Parts of Ten, or* $\frac{1}{10}$.	*Parts of a Hundred.*	*Parts of a Thousand.*	*Parts of Ten Thousands.*	*Parts of 100 Thousands.*	*Parts of a Million. &c.*

By this *Table* it is evident that as in whole Numbers or Integers, every Degree from the Units Place increases towards the Left-hand by a ten-fold Proportion: So in Decimal Parts every Degree decreases towards the Right-hand by the same Proportion, *viz.* by Tens.

Therefore these Decimal Parts or Fractions are really more *Homogeneal* or agreeing with Whole Numbers than Vulgar Fractions; for indeed all plain Numbers are in effect but Decimal Parts one to another.

That is, suppose any Series of equal Numbers, as 444, *&c.* The first 4 towards the Left is ten times the Value of the 4 in the Middle, and that 4 in the Middle is ten times the Value of the last 4 to the Right of it, and but the tenth Part of that 4 on the Left, *&c.*

Therefore all or any of them may be taken either as Integers, or Parts of an Integer: If Integers, then they must be set down without any Comma or separating Point betwixt them thus, 444. But if Integers, and one Part or Fraction, put a Comma betwixt them thus, 44,4 which signifies 44 whole Numbers, and 4 Tenths of an Unit: Again, if two Places of Parts be required, separate them with a Comma thus, 4, 44 *viz.* 4 Units, and 44 Hundred Parts of an Unit, *&c.*

From hence (duly compared with the Table) it will be easy to conceive that Decimal Parts take their Denomination from the Place of their last Figure.

$$\text{That is, } \left\{ \begin{array}{l} ,5 = \frac{5}{10} \\ ,56 = \frac{56}{100} \\ ,056 = \frac{56}{1000} \end{array} \right\} \text{ Parts of an Unit, \&c.}$$

Cyphers

Cyphers annex'd to Decimal Parts alter not their Value. As, ,50, and ,500, or ,5000, &c. are each but 5 Tenths of an Unit. For $\frac{50}{100}=\frac{5}{10}$. And $\frac{500}{1000}=\frac{5}{10}$. Or $\frac{5000}{10000}=\frac{5}{10}$. *Per Sect.* 4. of the last *Chapter*.

But Cyphers prefix'd to Decimal Parts decrease their Value, by removing them further from the Comma.

Thus,
- ,5= 5 Tenth Parts.
- ,05= 5 Parts of a Hundred.
- ,005= 5 Parts of a Thousand.
- ,0005= 5 Parts of Ten Thousands, &c.

Consequently the true Value of all Decimal Parts are known by their Distance from the Units Place; the which being once rightly understood, the rest will be easy.

Sect. 2. Addition *and* Subtraction *of* Decimals.

In setting down the proposed Numbers to be added, or subtracted, great Care must be taken in placing every Figure directly underneath those of the same Value, whether they be mix'd Numbers, or pure Decimal Parts, and to perform that, you must have a due Regard to the *Comma*'s, or separating Points, which ought always to stand in a direct Line one under another; And to the Right-hand of them carefully place the Decimal Parts, according to their respective Values or Distances from Unit. Then

Rule. *Add, or subtract them as if they were all* Whole Numbers; *And from their Sum or Difference, cut off so many* Decimal Parts *as are the most in any of the given* Numbers.

Examples in Addition.

Let it be required to find the Sum of these following Numbers, *viz.* 34,5+65,3+128,7+95+87,8+7,9, which being truly placed, will stand

Thus,

34,5
65,3
128,7
95,0
87,8
7,9

Their Sum required, 419,2

Example 2.

Let it be required to find the Sum of 25,854+34,578+9,076+13,907.

25,854
34,578
9,076
13,907
83,415 The Sum required.

When the Decimal Parts propos'd to be added (or ſubtracted) have not the ſame Number of Places, you may for Convenience of Operation ſupply or fill up the void Places, by annexing Cyphers. As in theſe *Examples*.

Example 3.	*Example* 4.	*Example* 5.
45,0700	574,678953	0,975642
50,7580	95,796430	,745257
123,0057	78,054600	,000598
74,7020	54,789000	,800700
24,8000	8,900000	,640530
318,3357	812,218983	3,162727

Examples in Subtraction.

Let it be required to find the *Difference* between 45,375 and 74,284.

	Example 1.		*Example* 2.		*Example* 3.
That is, From	74,284	From	437,5	From	75,0034
Take	45,375	Take	89,657	Take	57,875
Remains	28,909	*Remains*	347,843	*Remains*	17,1284

Example 4.

Let it be required to find the *Difference* between 562 and 93,5784.

	Example 4.		*Example* 5.
That is, From	562,	From	345,7578
Take	93,5784	Take	157,
the *Difference.*	468,4216	the *Difference*	188,7578

Note, The two laſt Examples are ſuppoſed to be ſupply'd with Cyphers, which if actually done would ſtand thus,

	562,0000		345,7578
	93,5784		157,0000
Remains	468,4216	As before,	188,7578

Example

Example 6.		*Example* 7.	
From	0,547893	From	1,000000
Take	0,439758	Take	0,997543
	0,108135		0,002457

The Proof of Addition, and Subtraction in Decimals is the ſame with that of Whole Numbers, *Page* 13, &c.

Sect. 3. Multiplication *of* Decimals.

Whether the Factors or Numbers to be multiplied are pure Decimals, or mix'd. Multiply them as if they were all Whole Numbers, and for the true Value of their Product obſerve this *Rule.*

Rule. *Cut off,* (viz. *ſeparate with a* Comma) *ſo many Places of* Decimal Parts *in the* Product, *as there are in both the* Factors *accounted together. As in theſe.*

Example 1.	*Example* 2.
3,024	32,12
2,23	24,3
9072	9636
6048	12848
6048	6424
6,74352	780,516

The Reaſon why ſuch a Number of Decimal Parts muſt be cut off in the Product, may be eaſily deduced from theſe Examples. Thus,

In *Example* 1. 'Tis evident, that 3 the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.* 3×2=6) So that of Neceſſity all the other Figures in the Product muſt be Decimal Parts; according as the *Rule* directs.

Or, the *Rule* is evident from the Multiplication of Whole Numbers only: Thus, ſuppoſe 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be ſo many Cyphers in the Product, as are in both the Factors. (*Vide Page* 18.) Now if inſtead of thoſe Cyphers in the Factors, we ſuppoſe the like Number of Decimal Parts; then it follows, that there ought to be the ſame Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the *Rule* may be otherwiſe made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and their Product

duct will be 780,516 as in *Example* 2. above. Now 32,12 = $32\frac{12}{100}$. and 24,3 = $24\frac{3}{10}$. which being brought into Improper Fractions (*per Sect.* 3. *Page* 50.) will become $32\frac{12}{100} = \frac{3212}{100}$. and $24\frac{3}{10} = \frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. *per Sect.* 7. *Page* 55.
But $\frac{780516}{1000} = 780\frac{516}{1000}$. *viz.* 780,516 as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid *Rule*, &c.

Example 3.

```
   78,546
      436
---------
   471276
  235638
 314184
---------
34246,056
```

Example 4.

```
    5745
   ,0675
--------
   28725
  40215
 34470
--------
387,7875
```

N. B. It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be Places of Decimal Parts by the *Rule*: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.

Example 5.

```
      ,2365
      ,2435
  ---------
      11825
     7095
    9460
   4730
  ---------
  ,05758775
```

Example 6.

```
    ,0347
    ,0236
---------
     2082
    1041
    694
---------
,00081892
```

When any proposed Number of Decimals is to be multiplied with 10. 100. 1000. 10000, &c. 'Tis only removing the separating Point in the Multiplicand, so many Places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, ,578 × 10 = 5,78. And, ,578 × 100 = 57,8
Again, ,578 × 1000 = 578. Or, ,578 × 10000 = 5780

These

These Things being consider'd, it will be easy to multiply Decimals, and determine their true Products. As in these following Examples.

57,056 Multiplied into 0,578 will produce 32,978368
7,6543 into 5,4246 will produce 41,52151578

0,56879 × 0,05674 = 0,0322731446
0,03246 × 0,02364 = 0,0007673544
87649 × 0,03687 = 3231,61863
94,35786 × 6,57869 = 620,7511100034
3,141592 × 52,7438 = 165,6995001296

Now it oftentimes happens, that it will be needless to express all the Figures of the Product at large (especially when the Factors have each of them many Places of Decimal Parts, as in the two last Examples) only so many of them as may suffice for the intended Design; and yet the Product may be as true to so many Figures as are retained, as if the Factors had been multiplied at large. And such compendious Contractions are not only of Curiosity, but may also be found of great Ease and Use to the ingenious Practitioner; especially in Resolving Adfected Æquations, or in calculating of Trigonometrical Problems by the *Natural Sines* and *Tangents*, &c. All which may be thus perform'd.

Viz. Set the Units Place of the Multiplier directly underneath that Figure of the Multiplicand, whose Place you intend to keep in the Product; And place all the other Figures of the Multiplier in a quite contrary Order to the usual Way. Then in Multiplying always begin at that Figure of the Multiplicand which stands over the Figure wherewith you are then a Multiplying, setting down the first Figure of each particular Product, directly underneath one another; yet herein you must have a due Regard to the Increase which would arise out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.

Example.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the proposed Numbers were to be multiplied at large, they must stand in a direct Order as usual.

Thus

Thus,	3,141592 52,7438	And would produce ten Places of Parts, as in the laſt Example.

But being 'tis required to have only four Places of thoſe Parts in the Product; ſet them down as above directed and they will ſtand

Thus 3,141592	The Multiplicand placed as before.
8347,25	The Multiplier in a reverſe Order.
1570796	The Product with 5, Regard had to 5 times 2.
62832	The Product with 2, increaſed with 9×2.
21991	Product with 7, increaſed with 5×7+9×7.
1257	Product with 4, increaſed with 1×4+5×4.
94	Product with 3, increaſed with 4×3.
25	Product with 8, increaſed with 4×8+1×8.
165,6995	The true Product as was required.

The Reaſon of this Contraction is very obvious from the whole Operation wrought at large.

Thus	3,141592
	52,7438
25	132736
94	24776
1256	6368
21991	144
62831	84
1570796	0
165,6995	001296

From hence 'tis evident that all the Figures in the Square to the Right-hand, are wholly omitted in the former Contraction; And that the laſt ſingle Product here, is the firſt there; conſequently the Reaſon of placing the Multiplier in a reverſe Order, muſt needs appear very plain.

Example 2.

Suppoſe it were required to multiply 257,356 with 76,48 and to have only the entire Product of Integers.

257,356
84,67
18015
1544
103
20
19682

The ſame at large.

	257,356
	76 48
20	58848
102	9424
1544	136
18014	92
19682,	58688

The chiefeſt Care and Difficulty that attends theſe Contractions, is the true ſetting down of the Units Place in the Multiplier underneath the proper Figure of the Multiplicand, according to the deſign'd Product.

Viz.

Viz. In *Example* 1. It was required to have four Places of Decimal Parts in the Product; therefore the Units Place of the Multiplier was ſet under the fourth Place of Decimals in the Multiplicand: And in *Example* 2, becauſe it was requir'd to have an entire Product of Integers only; therefore the Units Place of the Multiplier was ſet under the Units Place of the Multiplicand. This, I ſay, being once rightly underſtood, will render the Method eaſy in Practice.

Thom's Sect. 4. Diviſion *of* Decimals.

Diviſion is accounted the moſt difficult Part of Decimal Arithmetick; In order therefore to make it plain and eaſy, it will be convenient to reſume what has been ſaid in *Page* 25.

Viz. { *The* Quotient Figure *is always of the ſame* Value *or* Degree *with that Figure of the* Dividend, *under which the* Units *Place of its Product ſtands.*

As for Inſtance, Let 294 be divided by 4.

```
4) 294 (7   { This is not 7 but 70, becauſe the Units
   28         Place of 4 × 7 ſtands under the Tens Place
              of the Dividend.
   ——
    14 (3   But this is only 3.
    12
   ——
Remains (2)   Hence 73 2/4 is the Quotient.
```

Now if to the Remainder 2 there be annexed a Cypher, thus, 2,0 and then divided on, it muſt needs follow that the Units Place of the Product ariſing from the Diviſor into the Cuotient, will ſtand under the annexed Cypher; Conſequently the Quotient Figure will be of the ſame Value or Degree with the Place of that Cypher: But that's the next below the Units Place, therefore the Quotient Figure is of the next Degree or Place below Unity; that is, in the firſt Place of Decimal Parts.

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 is the true Quotient required.

This being well underſtood; Diviſion of Decimals may (in all the various Caſes) be eaſily perform'd. However, that it may be render'd plain and eaſy, even to the meaneſt Capacity, if poſſible; let Diviſion be again defin'd, as in *Page* 21.

Viz. If that Number which divides another, be Multiplied with the Number which is produced, their Product will be the Number divided.

This Definition alone (if compar'd with the Rule, *Page* 61) will afford a general Rule for discovering the true Value of the Quotient Figure in Division of Decimals.

Rule { *The Places of* Decimal Parts *in the* Divisor *and* Quotient, *being counted together, must always be equal in* Number *with those in the* Dividend. *And from this general* Rule *arise four Cases.*

Case 1. When the Places of Parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers.

As in these *Examples.*

```
8,45) 295,75 (35.          0,0078) ,4368 (56.
      2535                         390
      ----                         ---
      4225                         468
      4225                         468
      ----                         ---
      (o)                          (o)
```

Case 2. When the Places of Parts in the Dividend, exceed those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these *Examples.*

```
24,3) 780,516 (32,12                      436) 34246,056 (78,546
      729                                      3052
      ---                                      ----
       515                                      3726
       486                                      3488
       ---                                      ----
        291                                      2380
        243                                      2180
        ---                                      ----
         486     ,534) ,30438 (,57                2005
         486           2670                       1744
         ---           ----                       ----
         (o)            3738                       2616
                        3738                       2616
                        ----                       ----
                        (o)                        (o)
```

Case 3. When there are not so many Places of Parts in the Dividend, as are in the Divisor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in *Case* 1.

Examples.

Examples.

Let it be required to divide 192,1 by 7,684, And 441, by ,7875

```
7,684) 192,100 (25.            ,7875) 441,0000 (560.
        15368                          39375
        -----                          -----
        38420                          47250
        38420                          47250
        -----                          -----
         (0)                            (00)
```

Case 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their Defect by prefixing Cyphers to it.

Examples.

Let it be required to divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

```
      6699
      ----
      5550        Again  ,575) ,0007475 (,0013
      4785                       575
      ----                       ----
      7656                       1725
      7656                       1725
      ----                       ----
      (0)                         (0)
```

Note, When Decimal Numbers are to be Divided by 10. 100. 1000. 10000. *&c.* That is, when the Divisor is an Unit with Cyphers; Division is perform'd by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

Examples.

```
   10) 5784 (578,4          100) 5784 (57,84
 1000) 5784 (5,784        10000) 5784 (,5784
```

Note, *These Operations are the direct Converse to those in* Pa. 62.

I presume it's needless to give more Examples at large, only insert a few Dividends, and Divisors, with their Quotients; wherein are contained all the Varieties that can happen in Division of Decimals.

```
574) 493066 (859.            5,74) 49,3066 (8,59
574) 493,066 (,859           5,74) 493066,00 (85900.
574) 49,3066 (,0859         ,0574) 493,0660 (8590.
5,74) 4930,66 (859.         ,0574) ,463066 (8,59
```


There

There is alſo a compendious Way of contracting Diviſion; like unto that of Multiplication, *Page* 64. by which much Labour may be ſaved; eſpecially when the Diviſor hath many Places of Decimal Parts in it: And it's thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the firſt Figure in the Quotient will be: Then omit or prick off one Figure of the Diviſor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Diviſor; having a due Regard to the Increaſe which wou'd ariſe from the Figure ſo omitted.

Example.

Let it be required to divide 70,23 by 7,9863.

The Work contracted.

7,9863)	70,2300	(8,7938
....	63 8904	
	6 3396	
	5 5904	
	7492	
	7187	
	305	
	239	
	66	
	64	
	(2)	

The ſame at Large.

7,9863)	70,2300		(8,7938
	63 8904		
	6 3396	0	
	5 5904	1	
	7491	90	
	7187	67	
	304	230	
	239	589	
	64	6410	
	63	8904	
	0	7506	

The Work contracted I preſume is ſo obvious (if compared with the ſame at large) that it's needleſs to give any farther Explanation of it.

Sect. 5. *To* Reduce **Vulgar Fractions** *into* **Decimals**, *and the contrary.*

Any Vulgar Fraction being given it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

Rule. *Annex* Cyphers *to the* Numerator, *and then* divide *it by the* Denominator, *the* Quotient *will be the* Decimal Parts *equivalent to the given* Fraction; *or at leaſt ſo near it as may be thought neceſſary to approach.*

Example.

Example.

'Tis required to change or reduce $\frac{3}{4}$ into Decimals.

4) 3,00 (,75 The Decimal Parts required.

That is, $\frac{3}{4} = \frac{75}{100} = ,75$.

Again $\frac{1}{2} = ,5$ Thus 2) 1,0 (,5 And $\frac{1}{4} = .25$ 4) 1,00 (,25

Suppose it were required to change $\frac{4}{7}$ into Decimals.

7) 4,0000000000 (,5714285714 &c. $= \frac{4}{7}$

Note, When the last Figure of the Divisor (That is, the Denominator of the proposed Fraction) happens to be one of these Figures; viz. 1. 3. 7. or 9. (as in the last Example) then the Decimal Parts can never be precisely equal to the given Fraction; yet by continuing the Division on. you may bring them to be very near the Truth. As in this Example; Suppose it required to change $\frac{1}{13}$ into Decimal Parts.

13) 1,0000 (,0769230769230 7 &c. *ad infinitum.*

91

90
78

120
117

30
26

40
39

10
&c. As at first.

That is, 0,0769230769230 7 $= \frac{1}{13}$ *fere.*

And from hence it may be further observed; That in these imperfect Quotients, the Figures do return again and circulate in the same order as before: As you may easily perceive they begin to do in the seventh Place of both these last Examples.

These being understood, it will be easy to find the Decimal Parts equivalent to any known Part, or Parts of Coin, Weights, Measures, or Time, &c. If you first reduce the given Parts of Coin, *&c.* into a Vulgar Fraction, whose Denominator is the Number of those known Parts contained in the Integer, and the given Parts its Numerator.

Examples in Coin, *&c.*

1. Let it be required to find the Decimals of *16 s. 6 d.* First *16 s.* $= \frac{16}{20}$ of one Pound, and *6 d.* $= \frac{1}{40}$ of 1 *l.*

But $\frac{16}{20} + \frac{1}{40} = \frac{33}{40}$. Then 40) 33,000 (,825 the Decimal Parts required: That is, ,825 = 16 *s.* 6 *d.*

Again, Suppose it were required to find the Decimals equal to 3 *l.* 13 *s.* 4 *d.*

Here

Here 3*l.* is 3 *Integers*, and 13*s.* $= \frac{13}{20}$ of 1 *l.* and 4 *d.* $= \frac{4}{240}$. But $\frac{13}{20} + \frac{4}{240} = \frac{160}{240}$. Then 240) 160,000 (0,666666, *&c.* Hence 3 *l.* 13 *s.* 4 *d.* = 3,666666, *&c.* As was required.

2. What are the Decimals equal to 7¾ Inches, one Foot being made the Integer.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and ¾ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12} + \frac{3}{48} = \frac{31}{48}$. Then 48) 31,000 (,64583, *&c.* = 7¾ Inches.

3. Let it be required to change 8 ℥. 19 *Pwt.* 8 *Grains* into Decimals; one ℔ *Troy* being the Integer.

These being reduced into their least Terms, and added together will become $\frac{4304}{5760}$ of 1 ℔.

Then 5760) 4304,000 (,74722, *&c.* The Decimals required.

And thus may any proposed Parts of Coin, Weights, Measures, *&c.* be reduced or changed into Decimal Parts; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to reduce them almost mentally; or with the help of a very few Figures; without the Use of such large Tables as are usually inserted in Books of Decimal Arithmetick, or at most they may be contracted into such as these following; which if duly applied to those Tables in *Chap.* 3. will be found very useful.

Decimal Tables.

In English Coin.

0,05 . . . . . . = 1 *s.*
0,00416667 = 1 *d.*
0,00104167 = 1 *Farthing.*
1, *l.* being the *Integer.*

Troy Weight.

0,05 . . . . . . = *Pwt.*
0,00208333 = 1 *Grain.*
1, ℥. being the *Integer.*

Apothecaries Weights.

0,125 . . . . = 1 *Drachm.*
0,04166667 = 1 ℈
0,00208333 = 1 *Grain.*
1, ℥. being the *Integer.*

Averdupois Weight.

0,0625 . . . . = 1 *Ounce.*
0,00390625 = 1 *Drachm.*
1, ℔ being the *Integer.*

Averdupois Great Weight.

0 25 . . . . . . = ¼ *C.*
0,00892857 = 1 ℔.
0,00055803 = 1 *Ounce.*
1. *C.* being the *Integer.*

Time.

0,04166667 = 1 *Hour.*
0,00069444 = 1 *Minute.*
0,00001159 = 1 *Second.*
1, *Day* or 24 *Hours* being made the *Integer.*

The Use of these *Tables* will be evident by the following.

Example.

Example.

Let it be required to find the Decimal Parts equivalent to 17 s. 9 d. 2 *Farthings.*

First, 0,05=1 s. Therefore 17×,05=,85 .... =17 s.
And ,004166=1 d. Therefore ,004166×9=,037494=9 d.
And ,00104167=1 *Farth.* Also 2×,00104167 (=,002083=$\frac{1}{2}$ d.
Consequently their Sum, *viz.* 0,889577= 17 s. 9$\frac{1}{2}$ d.

Now to find the Value of Decimals, in known Parts of Coin, or Weights, &c. is only the Converse of the former Work. And is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is, Multiply the Decimals with such a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of: And the Product will be the Number required.

Example.

1. What is the Value of 0,825 Decimals of 1 Pound *Sterling.* That is, how many Shillings, Pence, &c.=,825. First, the next lower Denomination is 20, because 20 s. make one Pound.

Therefore 0,825
20
Shillings 16,500 And Parts of 1 *Shilling.*
12
Pence 6.000 Answer 0,825=16 s. 6 d.

Again, What are the known Parts of *English* Coin equal to 3,666666 Decimals.

Here the 3 Integers are 3 Pounds, Then ,666666
20
Shillings 13,333320
12
666640
333320
Pence 3.999840

Answer 3,666666=3 l. 13 s. 4 d. because the Fraction is so large one Penny more is added.

. What is the Value of 0,74722 Parts of 1 ℔. *Troy.*

First, ,74722
12
1 49444
7 4722
℥, 8,96664

Then, ,96664
20
Pwts. 19,33280

Again, ,33280
24
1 3312
6656
Grains 7,98720

	℥.	*Pwt.*	*Gr.*	
These collected are	8.	19.	8.	very near.

And

And thus any propofed Number of Decimals may be turn'd or chang'd into the known Parts of what they reprefent. *viz.* Whether they be Parts of Coin, Weights, Meafures, or Time, *&c.*

I have omitted inferting more Examples of this kind, becaufe I take the Excellency, and indeed the chief Ufe of Decimal Fractions to confift more in Geometrical Computations, than in the Common or Practical Parts of Arithmetick (as will appear further on) although even in thofe they are very ufeful upon feveral Accounts; efpecially in the Computations of Intereft and Annuities, *&c.* (But of that more in its proper Place.) I fhall therefore conclude this Chapter with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be Whole or Mix'd) be multiplied with a Fraction either Vulgar or Decimal, the Product will be lefs than the Multiplicand, in fuch a Proportion, as the Multiplying Fraction is lefs than an Unit or 1.

That is, *As the* Denominator *of the* Fraction *is to its* Numerator, *fo will the given* Number *be to the* Product.

Therefore, whenever any Number is to be multiplied with a Fraction, whofe Numerator is an Unit. Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, *&c.*

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend; by fuch a Proportion as Unity is greater than the Dividing Fraction.

Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, *&c.* But the Truth of thefe will be beft underftood after the next *Chapter.*

CHAP. VI.

Of Continued Proportions, *and how to* Change *or Vary the Order of Things.*

Sect. 1. *Concerning Arithmetical Progreffion, ufually called* Arithmetical Proportion Continued.

WHEN any Rank or Series of Numbers do either Increafe or Decreafe by an Equal Interval or Common Difference; thofe Numbers are faid to be in *Arithmetical Progreffion.*

As

As. $\begin{cases} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \text{\&c.} \\ 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 ; \end{cases}$ Here the Interval or Common Difference is 1.

Or, $\begin{cases} 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot \text{\&c.} \\ 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot \text{\&c.} \end{cases}$ Here the Common Difference is 2.

And so of any other Series, whose Common Difference is 3 . 4 . 5 . &c.

Lemma 1.

"If any three Numbers be in Arithmetical Progression; the "Sum of the two Extremes (*viz.* the first and last) will be equal "to the Double of the Mean or middle Number.

As in these 2 . 4 . 6 . Or 3 . 6 . 9 . Or 3 . 7 . 11.
Viz. $2+6=4+4$. Or $3+9=6+6$. And $3+11=7+7$. &c.

Lemma 2.

"If any four Numbers are in Arithmetical Progression, the Sum "of the two Extremes will be equal to the Sum of the two Means.

As in these. 2 . 4 . 6 . 8. Or 3 . 6 . 9 . 12.
Viz. $2+8=4+6$. And $3+12=6+9$. &c.

Corollary 1.

'From these two *Lemma's* 'tis easy to conceive; that if never 'so many Numbers be in Arithmetical Progression, the Sum of the 'two Extremes will be equal to the Sum of any two Means, that 'are equally distant from those Extremes.

As in these, 2 . 4 . 6 . 8 . 10 . 12 . 14 . 16.
Then $2+16=4+14=6+12=8+10$.

Or if the Number of Terms be odd as these.
2 . 4 . 6 . 8 . 10 . 12 . 14 . 16 . 18 . &c.
Then $2+18=4+16=6+14=8+12=10+10$.

Lemma 3.

"Every Series of Numbers in Arithmetical Progression is "composed of the Interval or Common Difference, so often re- "peated as there are Terms in the Progression except the First.

As in these, 1. 3. 5. 7. 9. 11. 13. 15. 17. &c.
Here the Interval or common Difference being two, it will be $1+2=3$. $3+2=5$. $5+2=7$. $7+2=9$. $9+2=11$. $11+2=13$. $13+2=15$. $15+2=17$, &c.

Corollary 2.

'Hence 'tis evident, that the Difference betwixt the two Ex- 'tremes (*viz.* 1 and 17) is composed of the common Difference, 'multiplied into the Number of all the Terms excepting the first.

As in the aforesaid Progression, 1. 3. 5. 7. 9. 11. 13. 15. 17.

The Number of Terms without the First is 8 } Multiply
The Common Difference is 2 }

The Difference betwixt the two Extremes 16

Proposition 1.

In any *Series* of *Numbers* in *Arithmetical Progression*, the two *Extremes*, and the *Number* of *Terms* being given; thence to find the *Sum* of all the *Series*.

Theorem. { *Multiply the Sum of the two Extremes into the Number of all the Terms; and divide the Product by 2. The Quotient will be the Sum of all that Series.* Per Corol. 1.

Example 1.

'Tis required to find the Number of all the Strokes a Clock strikes in one whole Revolution of the Index, *viz.* in twelve Hours.

Here 1+12=13 the Sum of the two Extremes.
And 12 the Number of all the Terms.

26
13

Then 2) 156 (78. The Number of Strokes required.

Example 2.

Suppose one Hundred Eggs were placed in a right Line a Yard distant from one another; and the first Egg were a Yard from a Basket; whether may a Man gather up those 100 Eggs singly one after another, still returning with every Egg to the Basket and put it in, before another Man can run four Miles. That is, which will run the greater Number of Yards.

In this Question 200+2=202 Is the Sum of the two Extremes.
And 100 Is the Number of all the Terms.

Then 2) 20200 (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles=7040 Yards } The Yards he runs that takes up
But 10100—7040=3060 } the Eggs more than the other.

Proposition 2.

In any *Series* of *Numbers* in *Arithmetical Progression*, the two *Extremes* and *Number* of *Terms* being given; thence to find the Common Difference of all the *Terms* in that *Series*.

Theorem. { *The Difference betwixt the two Extremes, being divided by the Number of Terms less an Unit or* 1. *The Quotient will be the common Difference of the Series.* Per Corol. 2.

Example.

Example 1.

One had twelve Children that differed alike in all their Ages; the youngeſt was nine Years old, the eldeſt was thirty ſix and a half; what was the Difference of their Ages, and the Age of each?

Here $36,5 - 9 = 27,5$ The Difference of the two Extremes.
And $12 - 1 = 11$. The Number of Terms leſs an Unit.
Then $11)\ 27,5\ (2,5$. The common Difference required.
Conſequently $9 + 2,5 = 11,5$ The Age of the youngeſt but one.
And $11,5 + 2,5 = 14$ The Age of the youngeſt but two. And ſo on for the reſt. *Per Corol.* 2.

Example 2.

A Debt is to be diſcharged at Eleven ſeveral Payments to be made in Arithmetical Progreſſion. The firſt Payment to be twelve Pounds ten Shillings, and the laſt to be ſixty three Pounds. What's the whole Debt, and what muſt each Payment be?

Per Theorem 1. Find the whole Debt thus:
$12,5 + 63 = 75,5$ The Sum of the Extremes.
11 The Number of Terms.

755
755

2) 830,5 (415,25 = 415 *l.* 5 *s.* The whole Debt.

Then *per Theorem* 2. Find the common Difference of each Payment.
Thus $63 - 12,5 = 50,5$ The Difference of the Extremes.
And $11 - 1 = 10$ The Number of Terms leſs 1.
Then 10) 50,5 (5,05 = 5 *l.* 1 *s.* The common Difference.
Conſequently 12 *l.* 10 *s.* + 5 *l.* 1 *s.* = 17 *l.* 11 *s.* The ſecond Payment.
And 17 *l.* 11 *s.* + 5 *l.* 1 *s.* = 22 *l.* 12 *s.* The third Payment, &c.

Example 3.

A Man is to travel from *London* to a certain Place in ten Days; and to go but two Miles the firſt Day, encreaſing every Day's Journey by an equal Exceſs; ſo that the laſt Day's Journey may be twenty nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to diſtant from *London*?

First $29-2=27$ The Difference of the Extremes.
And $10-1=9$ The Number of Terms less 1.
Then $9)\ 27\ (3$. The common Difference.
Consequently $2+3=5$. The second Day's Journey.
And $5+3=8$. The third Day's Journey, *&c.*
Again $29+2=31$ The Sum of the Extremes.
10 The Number of Terms.
$2)\ 310\ (155$ The Distance required.

There are eighteen *Theorems* more relating to Questions in Arithmetical Progression; but because they would require a great many Words to shew the Reason of them: I therefore refer the Reader to the second Part, *viz.* That of *Algebra*, where he may find their *Analytical Investigation.*

Sect. 2. Concerning Geometrical Proportion *Continued*; sometimes called *Geometrical Progression.*

When a Rank or Series of Numbers do either increase by one common Multiplicator, or decrease by one common Divisor; those Numbers are said to be in Geometrical Proportion continu'd.

As $\begin{cases} 2 . 4 . 8 . 16 . 32. \text{ \&c. here 2 is the common Multiplier.} \\ 64 . 32 . 16 . 8 . 4 . \text{ \&c. here 2 is the common Divisor.} \end{cases}$

Or $\begin{cases} 2 . 6 . 18 . 54 . 162 . \text{ \&c. here 3 is the common Multiplier.} \\ 162 . 54 . 18 . 6 . 2 . \text{ here 3 is the common Divisor.} \end{cases}$

Note, the common Multiplier (or Divisor) is called the *Ratio*; and it shews the Habitude or Relation the Numbers have to one another, *viz.* whether they are double, triple, quadruple, &c. Which *Euclid* thus defines.

Ratio (*or* Rate) *is the mutual Habitude or Respect of two* Magnitudes (*consequently two Numbers*) *of the same kind each to other, according to Quantity.* Eu. 5. Def. 3.

Proportion (rather *Proportionality* or *Analogy*) is a Similitude of *Ratio's*. Eu. 5. Def. 4.

So that there cannot be less than three Terms to form a Proportionality or Similitude of *Ratio's*; and if but three Terms, the second must supply the Place of two. As in these 2 . 4 . 8. That is. 2 : 4 :: 4 : 8 . (of :: see *Page* 5.)

Here 4 the middle Term supplies the Place of two Terms, *to wit*, of the second and third; 8 bearing the same Reason,

Likeness

Likeneſs or Proportion to 4. As 4 doth to 2. *Viz.* As 2 : is to 4 : : So is 4 : to 8.

Lemma 1.

If three Numbers are proportional, the Rectangle or Product of the two Extremes; viz. *of the firſt and laſt Terms, will be equal to the Square of the Mean or Middle Term.* (20 Eucl. 7.)

As in theſe 2 : 4 : : 4 : 8 Here $8 \times 2 = 16$ the Product of the Extremes.

And $4 \times 4 = 16$ the Square of the Mean: *Ergo* $8 \times 2 = 4 \times 4$.

Corol. 1.

' Hence it follows, that if the Product of any two Numbers be ' equal to the Square of a third Number; thoſe three Numbers ' will be in Proportion.

Lemma 2.

If four Numbers are proportional, the Product of the two Extremes, will be equal to the Product of the two Means. (19 Euclid 7.)

As in theſe, 2 : 4 : : 8 : 16. Here $16 \times 2 = 32$.
And $8 \times 4 = 32$. Conſequently $16 \times 2 = 8 \times 4$.

Corol. 2.

' From hence it follows, that if the Product of any two Num- ' bers, be equal to the Product of any other two Numbers, thoſe ' four Numbers are Proportionals.

And from theſe two *Lemma*'s, it will be eaſy to conceive, that if never ſo many Numbers are in continued Proportion; the Product of the two Extremes, will be equal to the Product of any two Means, that are equally diſtant from the Extremes.

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . &c.

Here $64 \times 2 = 32 \times 4 = 16 \times 8$. &c. And if the Number of Terms be odd,

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . 128 . &c.
Then $128 \times 2 = 64 \times 4 = 32 \times 8 = 16 \times 16$.

Note, The Character made uſe of to ſignify continued Proportionals is ÷÷

In

In every Series of $\div$ (viz. of continued Proportionals) that Number which is compared to another, is called the Antecedent of the *Ratio*; and that Number to which it is compared, is called its Consequent.

As in these, 2 : 4 :: 4 : 8. Here 2 is the Antecedent, and 4 is the Consequent; and 4 the middle Term is an Antecedent to 8 its Consequent: Whence it follows, that in every Series of $\div$ all the middle Terms between the first and last are both Antecedents and Consequents.

As in these, 2 . 4 . 8 . 16 . 32 . 64. &c. Here 4 . 8 . 16 32 are both Consequents and Antecedents.

For 2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 &c.

So that all the Terms except the last are Antecedents. And all the Terms except the first are Consequents.

Lemma 3.

If never so many Numbers are proportional, it will be: As any one of the Antecedents is to its Consequent: So will the Sum of the Antecedents be; To the Sum of all the Consequents. (12 Euclid 5.)

That is, in the foregoing Series.

$2 : 4 :: 2+4+8+16+32 : 4+8+16+32+64$.

For 'tis evident, that $4+8+16+32+64$ the Sum of all the Consequents, is double to $2+4+8+16+32$ the Sum of all the Antecedents; As 4 is to 2, according to the *Ratio*, and would have been triple, or quadruple, *&c.* had the *Ratio* been 3 or 4, *&c.*

Note, In every Series of $\div$ increasing the *Ratio* is found by dividing any of the Consequents by its Antecedent.

As in these, 2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162

Here 2) 6 (3 the *Ratio*. Or 6) 18 (3 *&c.*

From the second and third *Lemma*'s may be raised two general *Theorems* or *Rules*. for finding the Sum of any Series in $\div$ without a continued Addition of all the Terms.

Let the Series 2 . 4 . 8 . 16. 32 . 64 . 128 . be given to find its Sum.

Suppose z = the Sum of all the Terms.
Then $z-128$ = the Sum of all the Antecedents.
And $z-2$ = the Sum of all the Consequents.
But $2 : 4 :: z-128 : z-2$. *per Lemma* 3.
Ergo $4z-512=2z-4$. *per Lemma* 2.

Conse-

Consequently $4z-2z=512-4$.

Theorem. $z=\frac{512-4}{4-2}$ In Words at Length thus.

Theorem 1. *From the Product of the second and last Terms, subtract the Square of the first Term; that Remainder being divided by the second Term less the first; will give the Sum of all the Series.*

Or if the first Term, the common *Ratio*, and the last Term be only given. Then

Theorem 2. *Multiply the last Term into the* Ratio, *and from their Product subtract the first Term; divide that Remainder by the* Ratio *less Unity or* 1, *and it will give the Sum of all the Series.*

For $4z-2z=512-4$. As above.
Consequently $2z-z=256-2$ viz. the last divided by 2.

Then $z=\frac{256-2}{2-1}$ *Theorem* 2.

Example. Let 2 . 6 . 18 . 54 . 162 . 486 . be the given Series. Here 2 is the first Term, 3 is the *Ratio*, and 486 the last Term.

But $486\times3=1458$. And $1458-2=1456$.
Then $3-1=2$) 1456 (728 the Sum required.
That is $728=2+6+18+54+162+486$.

In either of these *Theorems* it is required to have the last Term known (the which in a long Series of ∺ will be very tedious to come at by a continued Multiplication, *&c.* It will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned; without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in Arithmetical Progression, and those in Geometrical Proportion.

If in any Series of Numbers in ∺ when the first Term is not an Unit or 1, there be assigned a Series of Numbers in Arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. Called *Indices* or *Exponents*.

Thus, $\begin{cases} 1 . 2 . 3 . 4 . 5 . 6 . 7 & \textit{Indices.} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128 & \textit{\&c.} ∺ \end{cases}$

Then

Then will the Addition or Subtraction of any two of thoſe Indices (or Numbers in Arithmetical Progreſſion) directly correſpond with the Product, or Quotient of their reſpective Terms, in the Series of ∺.

That is, $\left\{\begin{array}{l}\text{As } 3+4=7\\ \text{So } 8\times16=128\end{array}\right.$ the ſeventh Term in ∺.

Again, $\left\{\begin{array}{l}\text{As } 6+4=10.\\ \text{So } 64\times16=1024.\end{array}\right.$ The tenth Term in ∺

Or, $\left\{\begin{array}{l}\text{As } 7-3=4.\\ \text{So } 128\div8=16.\end{array}\right\}$ Or, $\left\{\begin{array}{l}\text{As } 6-2=4\\ \text{So } 64\div4=16\end{array}\right\}$ &c.

But if the Series of ∺ begin with an Unit, the Indices muſt begin with a Cypher.

As in theſe, $\left\{\begin{array}{l}0\ .\ 1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6.\ \&c.\\ 1\ .\ 2\ .\ 4\ .\ 8\ .16\ .32\ .\ 64.\end{array}\right.$

Now by the Help of theſe Indices, and a few of the firſt Terms in any Series of ∺. It is plain that any Term whoſe Place or Diſtance from the firſt Term is aſſigned, may be ſpeedily obtained without producing the whole Series.

Example 1.

A Man bought a Horſe, and was to give a Farthing for the firſt Nail, two for the ſecond, four for the third, &c. In ∺ The Number of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What muſt he have paid for the Horſe?

Firſt $\left\{\begin{array}{l}0\ .\ 1\ .\ 2\ .\ 3\ .\ 4\ .\ 5.\ \textit{Indices.}\\ 1\ .\ 2\ .\ 4\ .\ 8\ .16\ .32.\ \textit{Farthings}\text{ in ∺}\end{array}\right.$

Then, $\left\{\begin{array}{l}5+5=10\\ 32\times32=1024\end{array}\right.$ And $\left\{\begin{array}{l}10+10=20\\ 1024\times1024=1048576\end{array}\right.$

Again, $\left\{\begin{array}{l}4+3=7\\ 16\times8=128\end{array}\right.$ Laſtly, $\left\{\begin{array}{l}20+7=27\\ 1048576\times128=134217728\end{array}\right.$

Which is here to be accounted the 28 and laſt Term. Becauſe the firſt Term in the Series is 1. Which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the laſt Nail, by it the common *Ratio* which is 2, and the firſt Term which is 1. may be found the Sum of all the Series. *Per Theorem* 2.

$$\begin{array}{r} 134217728 \\ 2 \\ \hline 268435456 \end{array}$$ From this Product subtract 1.

Viz. 268435456—1=268435455. Then 2—1=1 the Divisor.

Consequently 268435455 is the Sum of all the Series or Price of the Horse in Farthings; which being brought into Pounds, *&c.* (See *Page* 46.) will be 279620 *l.* 5 *s.* 3 *d.* 3 *qrs.*

Example 2.

A cunning Servant agreed with a Master (unskill'd in Numbers) to serve him eleven Years without any other Reward for his Service but the Produce of one *Wheat Corn* for the first Year; And that Product to be sow'd the second Year; and so on from Year to Year until the End of the Time. Allowing the Increase to be but in a ten-fold Proportion.

'Tis required to find the Sum of the whole Produce.

First $\begin{cases} 1 \;.\; 2 \;.\; 3 \;.\; 4 \;.\; 5 \;. & \text{Indices or Years.} \\ 10 \;.\; 100 \;.\; 1000 \;.\; 10000 \;.\; 100000 & \text{Wheat Corns in} \div \end{cases}$

Then $\begin{cases} \text{As } 4 + 2 = 6 \\ \text{So } 10000 \times 100 = 1000000. \end{cases}$ the 6th Year's Produce.

And $\begin{cases} 6 + 5 = 11 \\ 1000000 \times 100000 = 100000000000. \end{cases}$ The eleventh or last Year's Produce.

Then, (either by *Theorem* 1 or 2) the Sum of all the Series will be, 111111111110 Corns. Now it may be computed from *Page* 31 and 34, that 7680 Wheat Corns round and dry out of the Middle of the Ear, will fill a Statute Pint. If so;

Then 7680) 111111111110 (14467592 Pints, but 64 Pints are contained in a Bushel.

Therefore 64) 14467592 (226056$1\frac{1}{8}$ Bushels. Suppose it to be sold for 3 Shillings the Bushel;

Then $\begin{cases} 226056\frac{1}{8} \\ 3 \end{cases}$

Shillings $678168\frac{3}{8}$ = 33908 *l.* 8 *s.* $4\frac{1}{2}$ *d.* A very good Recompence for 11 Years Service.

There are several pretty Questions resolved by Numbers in Arithmetical Progression; and by those in $\div$ which the ingenious Learner will easily perceive hereafter; *viz.* When we come to the Solution of Questions, relating to Interest and Annuities, *&c.*

There is alſo a third Kind of Proportion, called *Muſical*; which being but of little or no common Uſe, I ſhall therefore give but a ſhort Account of it.

Muſical Proportion or Habitude is, when of three Numbers the firſt hath the ſame Proportion to the third; As the Difference between the firſt and ſecond hath to the Difference between the ſecond and third.

As in theſe, *6* . 8 . 12. *viz.* *6* : 12 : : 8—*6* : 12—8

If there are four Numbers in Muſical Proportion, the firſt will have the ſame Proportion to the fourth; As the Difference between the firſt and ſecond hath to the Difference between the third and fourth.

As in theſe 8 . 14 : 21 . 84.
Here 8 : 84 : : 14—8=*6* : 84—21=*63*.
That is, 8 : 84 : : *6* : *63*.

The Method of finding out Numbers in Muſical Proportion is beſt expreſſed by Letters; as ſhall be ſhewed in the Algebraic Part.

Sect. 3. *How to* Change *or* Vary *the Order of Things*, &c.

This being a Thing not treated of in any common Books of Arithmetick (that I have had the Opportunity of peruſing) made me think it would be acceptable to the young Learner to know how oft 'tis poſſible to vary or change the Order or Poſition of any propoſed Number of Things.

As how many ſeveral Changes may be rung upon any propoſed Number of Bells; or how many ſeveral Variations may be made of any determined Number of Letters; or any other things expoſed to be varied.

The Method of finding out the Number of Changes, is by a continual Multiplication of all the Terms in a Series of Arithmetical Progreſſionals; whoſe firſt Term, and common Difference is Unity or 1. And laſt Term the Number of Things propoſed to be varied, *viz.* $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$, *&c.* As will appear from what follows.

1. If the Things propoſed to be varied are only two, they admit of a double Poſition, as to Order of Place; And no more.

$$\text{Thus, } \left\{ \begin{matrix} 1 & . & 2 \\ 2 & . & 1 \end{matrix} \right\} = 2 = 1 \times 2.$$

2. And if three things are propoſed to be varied, they may be

be changed six several Ways; as to their Order of Places; and no more.

For beginning with 1 there will be	1 . 2 . 3 1 . 3 . 2
Next beginning with 2 there will be	2 . 1 . 3 2 . 3 . 1
Again, beginning with 3 it will be	3 . 1 . 2 3 . 2 . 1
Which in all make 6 or 3 times 2. *viz.*	1 X 2 X 3 = 6

3. Suppose four things are proposed to be varied; Then they will admit of 24 several Changes, as to their Order of different Places.

For beginning the Order with 1 it will be Here is six different Changes.	1 . 2 . 3 . 4 1 . 2 . 4 . 3 1 . 3 . 2 . 4 1 . 3 . 4 . 2 1 . 4 . 2 . 3 1 . 4 . 3 . 2

And for the same Reason there will be 6 different Changes when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is 24=1X2X3X4. And by this Method of Proceeding, it may be made evident, that 5 Things admit of 120 several Variations or Changes; and 6 Things of 720, &c. As in this following *Table.*

The Number of Things proposed to be varied.	*The Manner how their several Variations are produced.*	*The different Changes or Variations every one of the proposed Numbers can admit of.*
1	1X1	=1
2	1X2	=2
3	2X3	=6
4	6X4	=24
5	24X5	=120
6	120X6	=720
7	720X7	=5040
8	5040X8	=40320
9	40320X9	=362880
10	362880X10	=3628800
11	3628800X11	=39916800
12	39916800X12	=479001600

These may be thus continued on to any assigned Number. Suppose to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 several Variations.

From these Computations may be started several pretty, and indeed very strange Questions.

Examples.

Six Gentlemen that were travelling met together by chance at a certain Inn upon the Road, where they were so pleased with their Host, and each others Company, that in a Frolick they made a Contract to stay at that Place, so long as they, together with their Host, could sit every Day in a different Order or Position at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Host, will admit of 5040 different Positions; but 5040 being divided by $365\frac{1}{4}$ the Number of Days in one Year, will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, (That before the great Fire of *London*: which happen'd *Anno* 1666) there was 12 Bells in St. *Mary le Bows Church* in *Cheapside, London*. Suppose it were required to tell how many several Changes might have been rung upon those 12 Bells: and at a moderate Computation how long all those Changes would have been ringing but once over.

First, $1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12=479001600$ the Number of Changes.

Then supposing there might be rung 10 Changes in one Minute: *viz.* $12\times10=120$ Strokes in a Minute, which is 2 Strokes in a Second of Time; now according to that rate there must be allowed 47900160 Minutes to ring them once over in all their different Changes; *viz.* 10) 479001600 (47900160.

In one Year there is 365 Days, 5 Hours and 49 Minutes; which being reduced into Minutes, is 525949.

Then 525949) 47900160 (91 Years, and 26 Days.

So long would those 12 Bells have been continually ringing without any Intermission, before all their different Changes could have been truly rung but once over. 'Tis strange, and seems almost incredible, that a few Things should produce such Varieties.

But

But that which ſeems yet more ſtrange and ſurpriſing, yea, even impoſſible to thoſe who are not a little verſ'd in the Power of Numbers, is, that if two Bells more had been added to the aforeſaid 12, they would have advanc'd the Number of Changes (and conſequently the Time) beyond common Belief. For 14 Bells would require (at the ſame rate of Ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were poſſible to ring 24 Bells in Changes, and at the ſame Rate of 10 Changes in a Minute, which is 4 Strokes in one Second; they would require more than 117000000000000000 Years to ring them but once over in all their different Changes; as may eaſily be computed from the precedent *Table*.

CHAP VII.

Of Proportion Disjunct; *commonly called the* Golden Rule.

PRoportion Disjunct, *or the* Golden Rule, *is either Direct or Reciprocal, called* Inverſe. *And thoſe are both Simple and Compound.*

SECT. I.

A Direct Proportion is, when of four Numbers, the firſt beareth the ſame Ratio or Proportion to the Second; as the third doth to the fourth.

As in theſe 2 : 8 :: *6* : 24.

Conſequently, the greater the ſecond Term is, in reſpect to the firſt; the greater will the fourth Term be, in reſpect to the third.

That is, as 8 the ſecond Term, is 4 times greater than 2 the firſt Term: So is 24 the fourth Term, 4 times greater than *6* the third Term.

Whence it follows, that if four Numbers are in direct Proportion, the Product of the two Extremes, will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to *Lemma* 2. *Page* 77.

For As $2 : 2\times4 :: 6 : 6\times4$. Or As $3 : 3\times5 :: 6 : 6\times5$.
But $2\times6\times4 = 2\times4\times6$. Or $3\times6\times5 = 3\times5\times6$.

That is, the Product of the Extremes is equal to that of the Means.

Again;

Again, the leſs the ſecond Term is, in reſpect to the firſt; the leſs will the fourth Term be, in reſpect to the third.

As in theſe $18 : 6 :: 12 : 4$.

That is, $18 : 18 \div 3 :: 12 : 12 \div 3$.

But $18 \times 12 \div 3 = 18 \div 3 \times 12$. *Viz.* $18 \times 4 = 6 \times 12$.

Conſequently $2 . 8 . 6 . 24$. And $18 . 6 . 12 . 4$. are true Proportionals, *per Corol.* 2. *Page* 77.

From theſe Conſiderations comes the Invention of finding a fourth Number in Proportion to any three given Numbers. Whence 'tis called, *the Rule of Three.*

For if the ſecond Number multiplied into the third, be equal to the firſt multiplied into the fourth, it is eaſy to conceive, that if the Product of the ſecond and third be divided by the firſt, the Quotient muſt needs be the fourth Number. For if that Number which divides another, be multiplied into the Quotient produced by that Diviſion; their Product will be equal to the Number divided. See *Page* 21.

As in theſe $2 : 8 :: 6 : 24$. Here $8 \times 6 = 48 = 24 \times 2$.

But if $24 \times 2 = 48$. then will $48 \div 2 = 24$. Or $48 \div 24 = 2$.

Note, Any four Numbers in direct Proportion may be varied ſeveral Ways. As in theſe.

Viz. If $2 : 8 :: 6 : 24$. Then $2 : 6 :: 8 : 24$.

And $6 : 24 :: 2 : 8$. Or $24 : 6 :: 8 : 2$. *&c.*

Theſe Variations being well underſtood, will be of no ſmall Uſe in the true Stating of any Queſtion in this *Rule of Three.*

When three Numbers are given; and it is required to find a fourth Proportional; the greateſt Difficulty (if there be any) will be in the right ſtating the Queſtion, or abſtracting the Numbers out of the Words in the Queſtion, and placing them down in their proper Order.

Now this will be very eaſy if it be truly conſidered, that always two of the three given Terms, are only ſuppoſed, and aſſign or limit the Ratio or Proportion. The third moves the Queſtion; and the fourth gives the Anſwer.

As for Inſtance; If 3 Yards of Cloth coſt 9 Shillings: What will 6 Yards coſt at the ſame Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two ſuppoſed Numbers that imply the Rate; as appears by the Word [if], *viz.* if 3 Yards coſt 9 Shillings (then comes the Queſtion) What will 6 Yards coſt?

N. B. The Term which moves the Queſtion hath generally ſome of theſe Words before it; *viz.* **What will? How many? How far? How long? or, How much?**

Then

Then (carefully observe this; *viz.*) The first Term in the Supposition must always be of the same Kind and Denomination with that Term which moves the Question. And the Term sought will always be of the same Kind and Denomination with the second Term in the Supposition.

Thus, { *Yds.* 3 : *Shil.* 9 :: *Yds.* 6 : *Shil.* —— }

Then

All Questions in direct Proportion may be answered by three several *Theorems.*

Theorem 1. { *Multiply the second and third Terms together, and divide their Product by the first Term; the Quotient will be the Answer required.* }

	Yds.		*Shil.*		*Yds.*		*Shil.*	
Thus	3	:	9	::	6	:	18.	The *Answer.*
			6					

3) 54 (18 *Shillings.* { Because the second Term was *Shillings.* }

Theorem 2. { *Divide the second Term by the first, then Multiply the Quotient into the third Term; and the Product will be the Answer requir'd.* }

	Yds.		*Shil.*		*Yds.*		*Shil.*
	3	:	9	::	6	:	18.
Thus	3)		9 (=3.		Then 3×6=18, as before.		

Theorem 3. { *Divide the third Term by the first, then multiply that Quotient into the second Term, and their Product will be the Answer.* }

	Yds.		*Shil.*		*Yds.*		*Shil.*
	3	:	9	::	6	:	18.
Thus	3)		6. (=2.		And 9×2=18. as before.		

Here you see that all the three *Theorems* are equally true; but the first is more general, and usually practised. Yet the two last may be readily performed when either the second or third Term can be divided by the first; and will be found of singular Use in the *Rules of Fellowship,* &c. as will appear further on.

Quest.

Quest. 2. If 8 ℔ of Tobacco cost 14 Shillings; What will half a hundred Weight (*viz.* 56 ℔) cost at the same Rate?

Thus 8 ℔ : 14 *s.* : : 56 ℔ : 4 *l.* 18 *s.* The *Answer.*

```
          14
        ----
         224
          56
        ----
8)  784 (=98 s.=4 l. 18 s.
```

Or thus 8) 56 (=7. Then 14×7=98 *s.* as before.

Quest. 3. If 14 Shillings will buy 8 ℔ of Tobacco; How much will 4 *l.* 18 *s.* buy after the same Rate?

Stated thus, 14 *s.* 8 ℔ : : 4 *l.* 18 *s.*=98 *s.* : ———
Then 98×8=784. And 14) 784 (56 ℔. The *Answer.*

Quest. 4. If half a hundred Weight of Tobacco be worth 4 *l.* 18 *s.* How much may I buy for 14 Shillings at that Rate?

Stated thus, 4 *l.* 18 *s.*=98 *s.* : 56 ℔ : : 14 *s.* : ———
Then 56×14=784. And 98) 784 (8 ℔. The *Answer.*

Quest. 5. Suppose 4 *l.* 18 *s.* will buy 56 ℔ of Tobacco; What will 8 ℔ of the same Tobacco cost?

This Question is thus stated, 56 ℔ : 4 *l.* 18 *s.*=98 *s.* : : 8 ℔. ———
Then 98×8=784. And 56) 784 (14 *s.* The *Answer.*

Note, The three last Questions are only the second varied, being proposed purely to give an Instance how any Question in this *Rule of Three* may be varied, according to *Page* 86.

Quest. 6. What will $\frac{3}{4}$ of a Yard of Velvet cost, when the Price of 21 Yards and a half is worth 22 *l.* 10 *s.* 6 *d.* This *Question* truly stated will stand

Thus, 21 $\frac{1}{2}$ *Yds.* : 22 *l.* 10 *s.* 6 *d.* : : $\frac{3}{4}$. To the *Answer.*

Which may be found three several Ways; *viz.* by *Reduction*; by *Vulgar Fractions*; and by *Decimals.*

1. By Reduction. Bring the first and third Terms into one Denomination; *viz.* into Quarters, and reduce the second Term into its least Denomination, *per Sect.* 4. *Page* 42.

Thus 21 $\frac{1}{2}$=86 Quarters. And 22 *l.* 10 *s.* 6 *d.*=5406 Pence.
Then 86 : 5406 : : 3 : 15 *s.* 8$\frac{5}{8}\frac{2}{6}$ *d.* For 5406×3=16218.

And

And 86) 16218 (=188$\frac{50}{86}$ *d.* Then 188$\frac{50}{86}$ *Pence* = 15 *s.* 8 *d.* 2$\frac{14}{43}$ *Farthings*; the *Anſwer* required.

2. The ſame *Queſtion* ſtated in Vulgar Fractions will ſtand

Thus 21$\frac{1}{2}$=$\frac{43}{2}$: 22$\frac{21}{40}$=$\frac{901}{40}$ :: $\frac{3}{4}$: (See *Sect.* 3. *Page* 50.)

Then $\frac{901}{40}$×$\frac{3}{4}$=$\frac{2703}{160}$. And $\frac{43}{2}$) $\frac{2703}{160}$ (=$\frac{5406}{6880}$ *Page* 55, 56.

Theſe $\frac{5406}{6880}$ Parts of a Pound are brought into Shillings by multiplying the Numerator with 20, and dividing the Product by its Denominator, *&c.*

Thus 5406×20=108120. And 6880) 108120 (15 *s.*
And there remains 4920. Again 4920×12=59040.
Then 6880) 59040 (8 *d.* and $\frac{50}{86}$, *&c.* as before.
3. The ſame wrought by Decimal Fractions will be thus;
21$\frac{1}{2}$=21,5 22 *l.* 10 *s.* 6 *d.*=22,525 and $\frac{3}{4}$=0,75
Therefore 21,5 : 22,525 :: 0,75 : to the *Anſwer.*
Then 22,525×0,75=16,89375
And 21,5) 16,89375 (,7857 *l.*=15 *s.* 8 *d.* 2 *far.* $\frac{275}{1000}$.

Queſt. 7. If 2 *C.* 3 *qrs.* 21 ℔ of Sugar coſt 6 *l.* 1 *s.* 8 *d.* what will 12 *C.* 2 *qrs.* coſt at the ſame Rate?
That is, 2 *C.* 3 *qrs.* 21 ℔ : 6 *l.* 1 *s.* 8 *d.* :: 12 *C.* 2 *qr.* To what?

4	20	4
11 *qrs.*	121 *s.*	50 *qrs.*
28	12	28
88	250	1400 ℔
22	121	

Viz. 308+21=329 ℔ : 1460 *d.* :: 1400 ℔ : ———

Then 1460 × 1400 = 2044000. And 329) 2044000 (6212$\frac{3}{4}$ *d.* = 25 *l.* 17 *s.* 8$\frac{3}{4}$*d.* the *Anſwer* required.
The ſame *Queſtion* ſtated in Decimals will ſtand
Thus 2,9375 :: 6,0833 :: 12,5 : To the *Anſwer.*
Then 6,0833×12,5=76,04125 which being divided by 2,9375 will give 25 88663 *&c.* the *Anſwer* in Decimals, which brought into Coin, will be 25 *l* 17,8$\frac{3}{4}$ *d.* as before.
Note, When the firſt Term is an Unit or 1, the Queſtion is anſwered by Multiplication only.
Example. Suppoſe I give 5 Shillings 4 Pence for one Ounce of Silver, what muſt I pay for 32$\frac{1}{2}$ Ounces at the ſame Rate.
That is 1 ℥ : 5 *s.* 4 *d.* :: 32 $\frac{1}{2}$ ℥ : To *&c*
Which is beſt ſtated thus 1 : 64 *d.* :: 32 5 :

Then $32,5 \times 64 = 2080$ *d.* = 8 *l.* 13 *s.* 4 *d.* the Anſwer required. For 1 neither multiplies nor divides.

When the ſecond, or third Term is an Unit or 1 then the Queſtion is anſwered by Diviſion only. As in this *Example.*

If a Silver Tankard, weighing 21 Ounces, coſt 5 *l.* 19 *s.* What's that an Ounce?

Thus 21 *oz*: 5 *l.* 19 *s.* = 119 *s.* : : 1 : 5 *s.* 8 *d.* To the *Anſwer.*
That is 21) 119 (= 5 *s.* $\frac{14}{21}$ = 5 *s.* 8 *d.*

The Proof of all Queſtions in the *Rule of Three Direct*, may be eaſily conceiv'd from what hath been already ſaid; *viz.* That the Product of the firſt and fourth Terms, muſt always be equal to the Product of the ſecond and third Terms.

Or otherwiſe, by varying the Queſtion, as in the ſecond, third, fourth and fifth Queſtions.

I ſhall conclude this *Section* with inſerting a few Queſtions and their Anſwers; leaving their Work for the Learner's Practice.

Queſt. 1. What will the Carriage of 17 *C.* 3 *qrs.* 11 ℔ come to, at the Rate of 7 *s.* the Hundred.

Anſwer, 6 *l.* 4 *s.* 11$\frac{1}{4}$ *d.*

Queſt. 2. If 6 *l.* 4 *s.* 11$\frac{1}{4}$ *d.* be paid for the Carriage of 17 *C.* 3 *qrs.* 11 ℔; What was paid for the Carriage of 1 ℔?

Anſwer, 3 *Farthings.*

Queſt. 3. A Grocer bought 3 *C.* 1 *qr.* 14 ℔ Weight of Cloves, at the Rate of 2 *s.* 4 *d. per* ℔, and ſold them for 52 *l.* 14 *s.* Whether did he gain or loſe by the Bargain, and *how much?*

Anſwer, he gained 8 *l.* 12 *s.*

Queſt. 4. A Draper bought of a Merchant Eight Packs of Cloth; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was twenty ſix Yards; he gave after the Rate of four Pounds ſixteen Shillings for 6 Yards, What came the eight Packs to; and what was it worth *per Yard?*

Anſw. They came to 6656 *l.* And is worth 16 *s. per Yard.*

Queſt. 5. A Merchant bought 436 Yards of Broad Cloth for 8 *s.* 6 *d. per Yard.* And ſold it again for 10 *s.* 4 *d. per Yard.* What did he gain by 436 Yards?

Anſwer, he gain'd 39 *l.* 19 *s.* 4 *d.*

Queſt.

Quest. 6. A *Goldsmith* bought a Wedge of Gold, which weighed 14 lb 3 *oz.* 8 *pw.* for 514 *l.* 4 *s.* What did he pay *per Ounce*?

Answ. 3 *l. per Ounce.*

Quest. 7: What will 48 *oz.* 17 *pwt.* 20 *Grains* of Silver Plate come to, at the Rate of 5 *s.* 6 *d. per Ounce*?

Answ. 13 *l.* 8 *s.* 10 $\frac{3}{4}$ *d.*

Quest. 8. If in four Weeks one spend 13 *s.* 4 *d.* How long will 53 *l.* 6 *s.* last at that Rate?

Answ. 6 *Years*, 47 *Days*, 2 *Hours*, 24.

Quest. 9. What will the eighth Part of a Ship be worth; when the half is valued at 1015 *l.* 10 *s*?

Answ. 253 *l.* 17 *s.* 6 *d.*

Quest. 10. The Sun is said to perform one entire Revolution, (or three hundred and sixty Degrees) in the Space of three hundred sixty five Days, five Hours, forty eight Minutes, and fifty seven Seconds of Time, called a Tropical or Solar Year. How much doth it move in one Day?

Answ. $5\overset{\prime}{9} \cdot \overset{\prime\prime}{8} \cdot \overset{\prime\prime\prime}{19}$ &c.

Quest. 11. If $\frac{5}{8}$ of a Yard of Velvet cost $\frac{2}{3}$ of a Pound *Sterling*; what will $\frac{1}{16}$ of a Yard cost of the same Velvet, at that Rate.

Answ. $\frac{16}{240}$ = 1 *s.* 4 *d.*

Quest. 12. Suppose 2 *l.* and $\frac{3}{8}$ of $\frac{1}{3}$ of a Pound *Sterling* will buy 3 Yards and $\frac{2}{3}$ of $\frac{3}{5}$ of a Yard of Cloth; How much will $\frac{3}{4}$ of a Yard cost at that Rate?

Answ. $\frac{2255}{4896}$ of a Pound = 9 *s.* 4$\frac{1}{2}$ *d.*

Sect. 2. Of Reciprocal Proportion; *usually called the* Rule of Three Inverse.

Reciprocal Proportion is, when of four Numbers the Third (*viz.* that which moves the Question) beareth the same *Ratio* to the First; as the Second does to the Fourth.

Therefore, the less the third Term is, in respect to the First; The greater will the fourth Term be, in respect to the Second.

Example 1.

If sixteen Men can do a Piece of Work in six Days; How many Days must eight Men require to do the same Work, at the same Rate of Working.

Here 'tis plain that eight Men must needs have more Time than 16 Men to do the same Work. Consequently the

greater the third Term is, in respect to the First, the lesser will the fourth Term be, in respect to the Second.

Example 2. If 8 Men can do a Piece of Work in 12 Days; how many Days will 16 Men require to do the same Work. Here it is plain the fourth Term must be less than the Second, because 16 Men undoubtedly can do the same Work in less Time than 8 Men can.

From these Considerations, compared with those in Page 85. *'twill be easy to perceive whether the Terms of any proposed* Question *are in* Direct *or* Reciprocal Proportion.

For when according to the true Meaning or Design of any Question *in* Proportion. MORE requires MORE, *or* LESS requires LESS, *the* Terms *are in* Direct Proportion, *as in the last* Section.

But if MORE requires LESS, *or* LESS requires MORE (*as above*) *then the* Terms *will be in* Reciprocal Proportion.

The Manner of placing down the proposed Terms is the same in both Rules, *viz.* The first Term in the Supposition must be of the same Kind and Denomination with the third Term which moves the Question; and the Term sought must be of the same Kind and Denomination with the second Term in the Supposition. As in the two last Examples.

		Men		Days		Men		Days
Thus, in	*Example* 1.	16	:	6	::	8	:	——
	Example 2.	8	:	12	::	16	:	——

The Question being truly stated, observe this *Theorem.*

Theorem. { *Multiply the first and second Terms together, and divide their Product by the third Term, the Quotient will be the Answer required.*

Thus in the second *Example* $12 \times 8 = 96$.

Then 16) 96 (=6 Days the Answer required.

That is, 16 Men may do the same Work in 6 Days as 8 Men can do in 12 Days.

Now the Reason of this Operation, (and consequently of the *Theorem*) is grounded upon this Consideration; *viz.* If 8 Men require 12 Days to do the Work, 'tis plain that one Man would require 8 times 12 Days = 96 Days to do the same Work. but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Answer required, *viz.* 16) 96 (6 As before, &c.

Quest. 3. Suppose 800 Soldiers were besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of these Soldiers must depart the Garrison, that the same Victuals may serve the remaining Soldiers 5 Months.

The

The Question truly stated, will stand

Thus, { *Month Soldier Month Soldier.*
2 : 800 : : 5 : ———
2

5) 1600 (320 So many Soldiers may stay in the Garrison.

Consequently, 800 — 320 = 480 Soldiers that must go out of the Garrison, which is the Answer required.

Question 4. *A.* Borrowed of his Friend *B.* 250 *l.* for six Months promising to do him the like Kindness upon Demand: Some time after *B.* desires *A.* to lend him 400 *l.* the Question is how long *B.* may keep the 400 *l.* to be fully satisfied for his former Kindness to *A*?

Answer, 3 Months 21 Days.

Thus, 250 *l.* : 6 Months : : 400 *l.* : ———
6

400) 1500 (3 Months.
12
3
28 Days in one Month.

4) 84 (21 Days.

Question 5. If a Penny White Loaf ought to weigh eight Ounces *Troy* Weight, when Wheat is sold for six Shillings six Pence the Bushel, what must it weigh when Wheat is sold for four Shillings the Bushel.

Answer, 13 oz.

Thus 6 *s.* 6 *d.* = 78 *d.* : 8 *Oz.* : : 4 *s.* = 48 *d.* To the *Answer.*
8

48) 624 (13 ℥ the *Answer* required.
48
144
144
(0)

The Proof of this *Inverse Rule* is easily deduced from its Operations; *viz.* The Product of the first and second Terms, must be equal to the Product of the third and fourth Terms.

Note,

Note, Any Queſtion that falls under this Inverſe Rule or Reciprocal Proportion, may be ſo ſtated as to have its Terms in Direct Proportion; by only changing the Places of the firſt and third Terms in the Queſtion. Thus,

Queſtion 6. If a Field will feed eighteen Horſes for ſeven Weeks, how long will it feed Forty two Horſes at the ſame Rate of feeding.

Firſt, 18 Horſes : 7 Weeks : : 42 Horſes : 3 Weeks.

Here the Terms are ſtated Inverſely, as before.

Otherwiſe thus, 42 Horſes : 7 Weeks : : 18 Horſes : 3 Weeks.

Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The Anſwer required.

Sect. 3. *Of* Compound Proportion; *commonly called the* Double Rule of Three.

Compound Proportion (as 'tis here meant) is, when there are five Numbers given to find out a ſixth Proportional; and this is generally performed by a double Poſition; that is, by ſtating and working the Queſtion at two Operations, either in Direct, or Reciprocal Proportion, according as the Queſtion requires.

And therefore it's called the Double Golden Rule; *or* Double Rule of Three.

The *Double Rule Direct* is, when the ſixth Term, or Number ſought, is found by two Operations, both of them in Direct Proportion.

Example 1. If a hundred Pounds gain ſix Pounds Intereſt in twelve Months; how much will three hundred Pounds gain in nine Months; at the ſame rate?

First 100 *l.* : 6 *l.* : : 300 *l.* : 18 *l.*
6
———
100) 1800 (18 *l.* } The Intereſt of 300 *l.* for twelve Months.

Months *Months*
Then, 12 : 18 *l.* : : 9 : 13 *l.* 10 *s.*
9
———
12) 162 (13 *l.* 10 *s.* The *Anſwer* required.

I ſuppoſe the Learner will eaſily conceive the Reaſon of theſe two Operations. For, Firſt it's plain by direct Proportion, that if 100 *l.* gain 6 *l.* in twelve Months, 300 *l.* will gain 18 *l.* in the ſame Time, and at the ſame Rate.

And

And by the ſame Rule 'tis plain, that if 12 Months will produce or give 18 *l.* Intereſt for 300 *l.* then 9 Months muſt needs give $13\frac{1}{2}$ for the ſame Sum, *viz.* 300 *l.*

The *Double Rule of Three Inverſe* is, when the ſixth Term or Number ſought is found at two Operations, (as before). But one of them requires an Anſwer in Reciprocal Proportion.

Queſtion 2. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, how many Days will 21 Buſhels ſerve 16 Horſes at the ſame Rate of Feeding?

This Queſtion being parted into two Poſitions, the firſt will be thus:

If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, how many Days will 21 Buſhels ſerve them?

Here 'tis plain that 21 Buſhels will ſerve them longer than 6 Buſhels; therefore the firſt Poſition falls in Direct Proportion.

Thus, { *Buſh.* *Days.* *Buſh.* *Days.*
6 : 8 :: 21 : 28
8

6) 168 (28 Days.

That is, if 6 Buſhels will ſerve 4 Horſes 8 Days, 21 Buſhels will ſerve them 28 Days.

The next Poſition muſt be to find how long the ſaid 21 Buſhels will ſerve 16 Horſes at the ſame Rate of Feeding: 'Tis plain, that 21 Buſhels cannot ſerve 16 Horſes ſo many Days as they will ſerve 4 Horſes; therefore this ſecond Poſition falls in Reciprocal Proportion.

Thus, { *Horſes.* *Days.* *Horſes.* *Days.*
4 : 28 : : 16 : 7 the Anſwer required.

After the like Manner any Queſtion in the *Double Rule of Three* may be anſwered by two ſingle Poſitions, if Care be taken in ſtating them right, *viz.* Whether their Operations muſt be performed by the *Single Rule Direct*, or *Inverſe*.

But all Queſtions in this *Double Rule*, where five Numbers are propoſed to find a ſixth, may more eaſily and readily be anſwered by one *General Theorem*; which compriſed both the *Direct* and *Inverſe Rules*: without conſidering either of them, being deduced from the ſingle Operations before-going.

But firſt you muſt carefully note, that in all Queſtions of this Nature, three of the five propoſed Terms are always conditional and

and ſuppoſed; and that the other two move the Queſtion. As for Inſtance in *Example* 1.

Viz. If 100 *l.* will gain 6 *l.* in 12 Months: Theſe three Terms are only ſuppoſed or conditional. Then comes the Queſtion; What will 300 *l.* gain in 9 Months? Now, in order to raiſe the General Theorem, let us ſuppoſe, inſtead of Numbers, theſe Letters.

Viz. Let $\left\{ \begin{array}{l} P = 100. \text{ The Principal.} \\ T = 12. \text{ The Time.} \\ G = 6. \text{ The Gain.} \end{array} \right\}$ In the Suppoſition of any propoſed Queſtion.

And $\left\{ \begin{array}{l} p = 300. \text{ The Principal.} \\ t = 9. \text{ The Time.} \\ g = 13{,}5 \text{ The Gain.} \end{array} \right\}$ The Three Terms wherein the Queſtion lies.

Then $P : G :: p : \frac{Gp.}{P.} =$ The Product of the two Means divided by the firſt Extreme.

That is, $100 : 6 :: \frac{300 \times 6}{100} = 18.$ Which is the firſt Part of the Queſtion.

Then $T : \frac{Gp}{P} :: t : g$ Which is the ſecond Part of the Queſtion.

Viz. $12 : 18 :: 9 : 13{,}5$

Ergo $Tg = \frac{Gpt}{P}$ That is, the Product of the Extremes is equal to that of the Means.

Conſequently, $TgP = Gpt$. Is the *Theorem*.

This *Theorem* affords two *Rules* by which all Queſtions in this *Double Rule of Three*, or rather of five Numbers, may be reſolved; due Regard being had to the true placing down of the propoſed Terms, which muſt be thus:

Always place the three Conditional Terms in this Order; Let that Number which is the principal Cauſe of Gain, Loſs, or Action, &c. (*viz. P.*) be put in the firſt Place; That Number which denotes the Space of Time or Diſtance of Place, &c. (*viz. T.*) be put in the ſecond Place. And that Number which is the Gain, Loſs, or Action, &c. (*viz. G.*) be put in the third Place. Now according to theſe Directions the Conditional Terms of the laſt Queſtions will ſtand thus; *P. T. G.*

That done, place the other two Terms which move the Queſtion, underneath thoſe of the ſame Name.

Thus, $\left\{ \begin{array}{l} P \cdot T \cdot G. \\ p \cdot t \cdot \end{array} \right.$

Then

Then if the Blank or Term sought, fall under the third Place, as in this Question,

It will be $\left\{\frac{G\,p\,t}{P\,T}\right.=g.$ Which gives this *Rule*.

Rule 1. { *Multiply the three last Terms together for a Dividend; and the two first together for a Divisor, the Quotient arising from them will be the sixth Term.*

That is, in our proposed *Example* 1.
Thus $6\times300\times9=16200$ The Dividend.
And $100\times12=1200$ The Divisor.
Then 1200) 16200 ($13\frac{1}{2}$ The Answer. As before.

But if the Blank or Term sought fall under the first Place, then

It will be $\left\{\frac{T\,g\,P}{t\,G}\right.=p$

Or if the Blank fall under the second Place,

It will be $\left\{\frac{T\,g\,P}{G\,p}\right.=t$ Either of these give this *Rule*.

Rule 2. { *Multiply the first, second and last Terms together for a Dividend: And the other two together for a Divisor; the Quotient arising from them will be the sixth Term.*

And because our *Example* 2. falls under the Consideration both of Direct and *Reciprocal Proportion*, let it be here propos'd again.

Viz. If 6 Bushels of Oats will serve 4 Horses 8 Days; how many Days will 21 Bushels serve 16 Horses, *&c.*

If the Terms of this Question be placed down as before directed they will stand

	Horses	Days.	Bushels.	
Thus	4	8	6	Terms in the Supposition.
	16		21	

Here the Blank falls under the second Place, therefore it must be found by the second *Rule*.

Thus $4\times8\times21=672$ the Dividend.
And $16\times6=96$ the Divisor.
Then 96) 672 (7 the Answer as before.

O *Quest.*

Quest. 3. What Principal or Stock will gain 20 *l.* in 8 Months at 6 *per Cent. per Annum.*

Prin.	*Time.*	*Gain*	
100 .	12 .	6	Terms in the Suppoſition.
.	8 .	20	

In this Queſtion the Blank falls under the firſt Place, therefore it muſt be found by the ſecond *Rule.*

Thus $100 \times 12 \times 20 = 24000$ the Dividend.
And $8 \times 6 = 48$ the Diviſor.
Then 48) 24000 (500 *l.* the Anſwer required.

The Proof of all Queſtions in this Double Rule of Five Numbers, is beſt perform'd by varying the *Queſtion*; viz. by ſtating it in another Order, as in the laſt *Example*: Thus,

If 100 *l.* gain 6 *l.* in 12 Months, what will 500 *l.* gain in 8 Months?

The Anſwer to this Queſtion muſt be 20 *l.* if the Work of the laſt *Example* be true.

$$\text{Stated thus} \left\{ \begin{array}{ccc} \textit{Prin.} & \textit{Time.} & \textit{Gain.} \\ 100 . & 12 . & 6 \\ 500 . & 8 . & \end{array} \right\} \text{then } \textit{per Rule } 1.$$

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$
Then 1200) 24000 (20. the *Anſwer, &c.*

Quest. 4. If Two Men can do 12 Rods of Ditching in 6 Days; How many Rods may be done by 8 Men in 24 Days, at the ſame Rate of Working.

Anſw. 192 Rods.

Quest. 5. If the Carriage of 5 *C.* 3 *qr.* Weight, 150 Miles; coſt 3 *l.* 7 *s.* 4 *d.* What muſt be paid for the Carriage of 7 *C.* 2 *qr.* 25 ℔ Weight, 64 Miles; at the ſame Rate.

Anſw. 1 *l.* 18 *s.* 7 ½ *d.*

Quest. 6. If 8 Men deſerve 2 *l.* Wages for 5 Days Work, How much will 32 Men deſerve for 24 Days; at the ſame Rate.

Anſw. 38 *l.* 8 *s.*

Quest. 7. Suppoſe a Hundred Pounds would defray the Expences of five Men for twenty two Weeks and ſix Days. How long would twelve Men be in ſpending of one hundred and fifty Pounds, and at the ſame Rate.

Anſw. 14 Weeks and 2 Days.

CHAP.

CHAP. VIII.

Of Trading in Company; usually called the Rule of Fellowship; *Also* Bartering, *and* Exchanging *of Coins*, &c.

THE *Rule of Fellowship*, is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to his Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the Single and Double *Rules of Fellowship*.

Sect. 1. *The* Single Rule of Fellowship; viz. *That without* Time.

By the *Single Rule of Fellowship*, are adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore 'tis usually call'd the *Rule of Fellowship* without Time: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

'For; as the total Sum of Money in the Stock is in Proportion 'to the whole Gain or Loss; so is every Man's particular Part of 'that Stock; to his particular Share of that Gain, or Loss.

Quest. 1. There are three Partners, suppose *A*, *B*, and *C*, make a Joint-Stock of 96 *l.* in this Manner.

A, puts in 24 *l.* B, puts in 32 *l.* and *C*, puts in 40 *l.* with this 96 *l.* they trade and gain 12 *l.* 'Tis required to find each Man's true Part of that Gain.

The first Operation for *A*'s Part of the Gain will stand

Thus 96 *l* : 12 *l* : : 24 *l* : 3 *l.* = *A*'s Gain.
Secondly, 96 *l* ; 12 *l* : : 32 *l* : 4 *l.* = *B*'s Part of the Gain.
Again, 96 *l* : 12 *l* : : 40 *l* : 5 *l.* = *C*'s Part of the Gain.
Proof 3 *l.* + 4 *l.* + 5 *l.* = 12 *l.* the whole Gain.

That is, if the Sum of each Man's particular Gain, amount to the whole Gain, the Work is true; if not, some Error is committed which must be found out.

Note, These Operations will be very much abbreviated, if you work them by *Theorem* 2. *Page* 87. For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions,

Therefore $96 : 12 :: 1 : 0,125$ is a common Multiplicator.

Then	$24 \times 0,125 = 3$ *l.* for *A*'s Part		
And	$32 \times 0,125 = 4$ *l.* for *B*'s Part	}	As before.
Again	$40 \times 0,125 = 5$ *l.* for *C*'s Part		

Now this Method is more readily perform'd than the other, especially when the Partners are many; because one single Division serves for all the Work.

Quest. 2. Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tun of Wine: thus, *A* loaded 98 Tun, *B* 86 Tun, and *C* 64 Tun. By Extremity of Weather the Seamen were forc'd to cast or throw 93 Tun of it over-board. How much of this Loss must each Merchant sustain?

First	$248 : 93 :: 1 : 0,375$	the common Multiplier.
Then	$98 \times 0,375 = 36,75$	for *A*'s Loss.
And	$86 \times 0,375 = 32,25$	for *B*'s Loss.
Again	$64 \times 0,375 = 24,$	for *C*'s Loss.
Proof	$93,00 =$	the whole Loss.

Now if the Question were to find how much of the remaining Wine that was saved, belongs to *A*, to *B*, and to *C*.

Then	$98 - 36,75 = 61,25$	belongs to *A*.
And	$86 - 32,25 = 53,75$	belongs to *B*.
And	$64 - 24, = 40$	belongs to *C*.

That is: *A* ought to have 61 Tun and 63 Gallons. *B* ought to have 53 Tun and 189 Gallons. And *C* ought to have 40 Tun of what was left.

Quest. 3. Suppose six Men, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 *l.*

		l.	*s.*	Decimals.
Thus	*A* puts in	654	. 10	= 654,5
	B ——	543	. 15	= 543,75
	C ——	480	. 00	= 480,
	D ——	254	. 10	= 254,5
	E ——	365	. 05	= 365,25
	F ——	260	. 00	= 260,
	The whole Stock	2558	. 00	= 2558,00, according to the Question.

With

With this Stock of 2558 *l.* they trade eighteen Months, and gain 831 *l.* 7 *s.* 'Tis required to find every Man's Part or Share of that Gain.

Note, Altho' the Time of Trading, *viz.* Eighteen Months be mentioned in the Question; yet 'tis no Way concerned in answering of it; as you may observe by the following Work.

First, 2558 *l.* : 831,35 *l.* : : 1 *l.* : 0,325 Decimal Parts.
Consequently, 1 *l.* : 0,325 : : 654,5 : 212,7125. That is,

A's Stock	654,5 ×0,325=212,7125	for *A*.
B's Stock	543,75×0,325=176,71875	for *B*.
C's Stock	480, ×0,325=156,000	for *C*.
D's Stock	254,5 ×0,325= 82,7125	for *D*.
E's Stock	365,25×0,325=118,70625	for *E*.
F's Stock	260, ×0,325= 84,5	for *F*.

		l. parts	*l.*	*s.*	*d.*
That is,	*A* Gains	212,7125	=212	14	03
	B ——	176,71875	=176	14	04 ½
	C ——	156,0000	=156	00	00
	D ——	82,7125	= 82	14	03
	E ——	118,70625	=118	14	01 ½
	F ——	84,5	= 84	10	00
Proof.	Sum	831,35	=831	07	00 the Gain.

I have omitted resolving this Question according to the usual Method (as before directed) or finding every Man's particular Part of the Gain by the *Golden Rule*, as in the first Work of Example 1. leaving that for the Learner's Practice.

Sect. 2. *The* Double Rule *of* Fellowship; *or that with Time.*

This is usually called the *Double Rule of Fellowship*; because every particular Man's Money is to be considered with relation to the Time of its Continuance in the Joint-Stock.

Question 1. *A* and *B* join in Partnership upon these Terms, *viz.* *A* agrees to lay down 100 *l.* and to employ it in Trade 3 Months: Then *B* is to lay down his 100 *l.* and with the whole Stock of 200 *l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21 *l.* 'Tis required to know what each Man's Part of the Gain ought to be, according to his Stock, and the Time of imploying it.

Here

Here it is but reasonable to conclude, that *A.* ought to gain more than *B.* notwithstanding their Stocks of Money are equal; because *A.* imploy'd his Money a longer Time than *B.*

Now for solving of this Question, let us suppose *A*'s 100 *l.* employed the first 3 Months to gain $Z =$ a Sum as yet unknown; then it must gain $2Z$, in 6 Months; and to find what *B* must gain it will be,

l.	*Months*		
100 .	6 .	$2Z = A$'s Gain	per Rule 1. Page 97.
100 .	3 .	To *B*'s Gain	

Ergo $\frac{100 \times 3 \times 2Z}{100 \times 6} = B$'s Gain.

But *A*'s Gain added to *B*'s Gain must $= 21$ *l.* the whole Gain the Question,

Therefore $2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21$ *l.*

That is, $100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6$,

Which contracted is, $900 \times 2Z = 21 \times 600$.

Consequently $2Z = \frac{21 \times 600}{900}$ which gives the following *Analogy*,

Viz. $900 : 21 :: 600 : 2Z = 14$ *l.* for *A*'s Gain.

And $900 : 21 :: 100 \times 3 = 300 : 7$ *l.* for *B*'s Gain.

Now this Way of Arguing hath not only resolved the present Question; but it also affords (and demonstrates) a general Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule. *Multiply every particular Man's Stock, with the Time it is employed; then it will be, As the Sum of all those Products; Is to the whole Gain (or Loss). So is every one of those Products; to its proportional Part of that whole Gain (or Loss)*

Question 2. Three Merchants *A. B.* and *C.* enter into Partnership thus; *A.* puts into the Stock 65 *l.* for 8 Months; *B.* puts in 78 *l.* for 12 Months; and *C* puts in 84 *l.* for 6 Months. With these they traffick, and gain 166 *l.* 12 *s.* 'Tis required to find each Man's Share of the Gain, proportionable to his Stock and Time of employing it.

l. *A*'s

1. *A*'s Stock 65 *l.* × 8 Months, the Time it was employed = 520
2. *B*'s Stock 78 *l.* × 12 Months, the Time it was employed = 936
3. *C*'s Stock 84 *l.* × 6 Months, the Time it was employed = 504

The Sum of those Product is, 1960

Then, according to the Rule, the several Proportions will stand thus,

1960 : 166,6 :: 520 : 44,2 = 44 *l.* 4 *s.* 0 *d.* for *A.*
1960 : 166,6 :: 936 : 79 56 = 79 *l.* 11 *s.* $2\frac{1}{2}$ *d.* for *B.*
1960 : 166,6 :: 504 : 42,84 = 42 *l.* 16 *s.* $9\frac{1}{2}$ *d.* for *C.*

The whole Gain = 166 *l.* 12 *s.* 0 *d.*

Or you may work as in some of the former Examples, *viz.* by finding the proportional Part of the Gain due to one Pound, *&c.*

Thus, 1960 : 166,6 :: 1 : 0,085 the common Multiplier.

Then 520×0,085 = 44,2 for *A.*
And 936×0,085 = 79.56 for *B.*
And 504×0,085 = 42 84 for *C.*
&c. As before.

Question. 3. Six Merchants, viz. *A. B. C. D. E.* and *F.* enter into Partnership, and compose a Joint-Stock in this manner;

		l.	*s.*			
Viz.	*A* puts in	64	10	for	$4\frac{1}{2}$	Months
	B	78	15		6	
	C	100	00		$8\frac{1}{4}$	
	D	80	10		12	
	E	74	12		$9\frac{1}{2}$	
	F	125	15		7	

They traffick, and gain 258 *l.* 18 *s.* $4\frac{1}{2}$ *d.* 'Tis required to find every Man's Share of the Gain, according to his Stock and Time it was employed.

The several Stocks of Money, and their respective Times being first brought into Decimals, and then multiplied together will produce these following Products.

	l. *Months.*		
A's Stock	64.5 ×4,5	the Time it was employed	= 290,25
B's Stock	78.75×6,	the Time it was employed	= 472,5
C's Stock	100, ×8.25	the Time it was employed	= 825,0
D's Stock	80.5 ×12.	the Time it was employed	= 966,0
E's Stock	74.6 × 9,5	the Time it was employed	= 708.7
F's Stock	125,75×7,	the Time it was employed	= 880,25

The Sum of those Products = 4142,7

Then

Then if you work by the common way; it will be $4142,7 : 258,91875 :: 290,25 : 18,140625 = 18$ *l.* 2 *s* $9\frac{3}{4}$ *d.* for *A*'s Part of the Gain; and so on for the rest.

But if you work by the easiest Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus $4142,7 : 258,91875 :: 1 : 0,0625$

			l.	*s.*	*d.*	
Then	$290,25 \times 0,0625 = 18,140625$	=	18 .	2 .	$9\frac{3}{4}$	for *A*
And	$472,5 \times 0,0625 = 29,53125$	=	29 .	10 .	$7\frac{1}{2}$	for *B*
	$825, \times 0,0625 = 51,5625$	=	51 .	11 .	3	for *C*
	$966, \times 0,0625 = 60,375$	=	60 .	7 .	6	for *D*
	$708,7 \times 0,0625 = 44,29375$	=	44 .	5 .	$10\frac{1}{2}$	for *E*
	$880,25 \times 0,0625 = 55,015625$	=	55 .	0 .	$3\frac{3}{4}$	for *F*
	The whole Gain	=	258 .	18 .	$4\frac{1}{2}$	

These few Examples being well understood, are, I presume, sufficient to shew the whole Business of Fellowship, &c.

Sect. 3. *Of* Bartering.

When Merchants, or Tradesmen, exchange one Commodity for another, 'tis called *Bartering*; and the only Difficulty in this Way of Dealing, lies in the due proportioning the Commodities to be exchanged; so as that neither Party sustain Loss.

Question 1. Two Merchants, *A.* and *B.* *Barter*; *A.* would exchange 5 *C.* 3 *qrs.* 14 lb of *Pepper*, which is worth 3 *l.* 10 *s.* *per C.* with *B.* for *Cotton*, worth 10 *d.* *per* lb Weight. How much *Cotton* must *B.* give to *A.* for his *Pepper*?

Note, In order to the resolving of this Question, (and all other Questions of this Nature) you must first find, by the Rule of Three, (or otherwise) the true Value of that Commodity whose Quantity is given; (which in this Question is *Pepper*). And then find how much of the other Commodity will amount to that Sum, at the Rate proposed.

$$\left.\begin{array}{l}\text{First } 5\ C.\ 3\ qrs.\ 14\text{ lb} = 5,875 \\ \text{And } 3\ l.\ 10\ s.\ 0\ d. = 3,5\end{array}\right\}\text{in Decimals.}$$

Then $1 : 3,5 :: 5,875 : 20,5625 = 20$ *l.* 11 *s.* 3 *d.* the true Value of the *Pepper*.

Next, 'tis easy to conceive, that *A.* ought to have as much *Cotton* at 10 *d.* *per* lb. as will amount to 20 l. 11 *s.* 3 *d.* which may be thus found;

10 *d.* : 1 lb. :: 20 l. 11 *s.* 3 *d.* = 4235 *d.* : 493,5 lb.

That

That is, 4 *C.* 1 *qr.* 17 $\frac{1}{2}$ ℔ of Cotton. And ſo much *B* muſt give to *A* in Exchange for his 5 *C.* 3 *qrs.* 14 ℔ of Pepper.

Queſtion 2. Two Merchants *A.* and *B.* Barter thus, *A.* hath 86 Yards of broad Cloth worth 9 *s.* 2 *d. per* Yard, ready Money; but in Barter he will have 11 *s. per* Yard. *B.* hath Shalloon worth 2 *s.* 1 *d. per* Yard ready Money; 'Tis required to find how many Yards of the Shalloon *B.* muſt give to *A.* for his Cloth, to make his Gain in the Barter equal to that of *A*'s.

The Method of reſolving this, and the like Queſtions, differs a little from the laſt Caſe; for in this you muſt firſt find what Advance *B.* ought to make *per* Yard upon his Shalloon, in Proportion to what *A.* hath done upon a Yard of his Cloth.

Thus $\begin{cases} \textit{s. d.} \quad \textit{d.} \quad \textit{s.} \quad \textit{d.} \quad \textit{s. d.} \quad \textit{d.} \quad \textit{s. d.} \quad \textit{d.} \\ 9.\ 2 = 110 : 11 = 132 :: 2.\ 1 = 25 : 2.\ 6 = 30 \end{cases}$

the advanced Price for a Yard of *B*'s Shalloon. Then proceed as before in the laſt Example.

Thus 1 Yard : 11 *s.* :: 86 Yards : 946 *s.* = 47 *l.* 6 *s.* the advanced Value of all the Cloth.

Next, If 2 *s.* 6 *d.* will buy one Yard of Shalloon, at its advanced Price, how many Yards will 47 *l.* 6 *s.* buy?

Thus 2,5 : 1 :: 946 : 378,4 Yards.

That is, *B.* muſt give 378$\frac{2}{5}$ Yards of his Shalloon to *A.* for his 86 Yards of broad Cloth.

Theſe two Examples are ſufficient to ſhew the Learner, that the Method of Bartering, or Exchanging Commodities for Commodities, wholly depends upon a clear Underſtanding of the Golden Rule; which indeed is ſo called becauſe of its Univerſal Uſe.

Sect. 4. *Of* Exchanging Coins.

Exchanging the Coins of one Country for thoſe of another is like the Buſineſs of Bartering Commodities. That is, it conſiſts in finding what Sum of one Country Coin will be equal in Value to any propoſed Sum of another Country Coin. And in order to perform that, it will be very neceſſary to have a true Account at all Times of the juſt Values of thoſe Foreign Coins which are to be exchanged, as they are compared in Value with our *Engliſh* Coin.

I ſay, at all Times, becauſe the *Par* of Exchange (as the Merchants call it) differs almoſt every Day from *London* to other Countries. That is, it riſes and falls, according as Money is plenty or ſcarce; or according to the Time allowed for Payment of the Money in Exchange, *&c.*

P Thoſe

Those that desire to be fully satisfied in the common Values of Foreign Coins, may find them in a Book called the *Merchants Map of Commerce*, which for Brevity's sake I have omitted Transcribing, and only collected these few of Coin.

Foreign Coin.		English Coin. *l. s. d.*
French Coins.	A Denier =	0 . 0 . 0 $\frac{3}{40}$
	12 Deniers = 1 Soulz =	0 . 0 . 0 $\frac{9}{10}$
	20 Soulz = 1 Livre =	0 . 1 . 6
	3 Livres = 1 Crown =	0 . 4 . 6
Low-Country Coin.	A Stiver =	0 . 0 . 1 $\frac{1}{5}$
	6 Stivers = 1 Flemish Shilling =	0 . 0 . 7 $\frac{1}{5}$
	20 Stivers = 1 Gilder =	0 . 2 . 0
	10 Gilders = 33$\frac{1}{3}$ Shillings or a Flemish Pound =	1 . 0 . 0
	A Embdem Doller =	0 . 2 . 3 $\frac{3}{5}$
	A Campen Doller =	0 . 2 . 7 $\frac{1}{5}$
	A Zealand Doller =	0 . 3 . 0
	A Lyons Doller =	0 . 4 . 0
	A Specie Doller =	0 . 5 . 0
	Ducatoon =	0 . 6 . 3 $\frac{3}{5}$
Germany.	A Rix Doller of the Empire =	0 . 4 . 5 $\frac{1}{4}$
	A Gilder of Nuremberg =	0 . 7 . 1
In Italy and Spain.	The Livre at Leghorn =	0 . 0 . 9
	Florence Crown Courrant =	0 . 5 . 3
	Venice Ducat de Banco =	0 . 4 . 4
	The Currant Ducat =	0 . 3 . 4
	The Naple Ducat =	0 . 5 . 0
	The Cadiz Ducat =	0 . 5 . 6 $\frac{1}{4}$
	The Barcelona Ducat =	0 . 6 . 0
	The Valentia Ducat =	0 . 5 . 3
	The Bergonia Ducat =	0 . 4 . 4
	The Portugal Testoon =	0 . 1 . 3
	The Piece of Eight =	0 . 4 . 6

Note, The *English* generally reckon their Exchange with other Countries by Pence, *viz.* other Countries value their Crowns, Dollers, or Ducats, *&c.* by *English* Pence, except with some Parts of the *Low-Countries*, with whom the Exchange is in Pounds *Sterling*.

Quest. 1. How many Dollers at 4 *s.* 6 *d. per* Doller, may one have for 162 *l.* 18 *s.* *Answer*, 724 Dollers.

Thus

Thus 162 *l.* 18 *s.*=3258 *s.* and 4 *s.* 6 *d.*=4,5 *s.*
Then 4,5 : 1 : : 3258 : 724 the *Answer.*

Quest. How many *Saragosa* Ducats of 5 *s.* 6 *d.* the Ducat may be had for 275 *Bergonia* Ducats, at 4 *s.* 4 *d.* the Piece?
Answer, 216 and 3 *s.* 8 *d.* over.

Thus 5 *s.* 6 *d.*=66 *d.* and 4 *s.* 4 *d.*=52 *d.*
Then 275×52=14300 *d.*=275 Ducats.
Consequently 66) 14300 ($216\frac{2}{3}$ the *Answer* requir'd.

Quest. 3. A Traveller would change 233 *l.* 16 *s.* 8 *d.* *Sterling* Money; for *Venice* Ducats at 4 *s.* $9\frac{1}{2}$ *d. per* Ducat. How many Ducats must he have?
Answer, 976 Ducats.
Thus 4 *s.* $9\frac{1}{2}$ *d.*=57,5 *d.* and 233 *l.* 16 *s.* 8 *d.*=56120 *d.*
Then 57,5 *d.*) 56120 *d.* (976 the *Answer* required.

Quest. 4. A Cashier hath received 759 Ducats, at 7 *s.* 6 *d. per* Ducat; And 579 Dollers at 4 *s.* 8 *d. per* Doller: Which he would exchange for *Flemish* Marks at 14 *s.* 3 *d. per* Piece. How many ought he to have?
Answer, 589 Marks, and 15 *d.* over.

For 7 *s.* 6 *d.*=90 *d.* and 4 *s.* 8 *d.* = 56 *d.*
Then { 759×90=68310 *d.* the Value of the Ducats:
579×56=32424 *d.* the Value of the Dollers.

their Sum =100734 *d.*

And 14 *s.* 3 *d.*=171 *d.* the *Flemish* Mark in Pence.
Consequently 171) 100734 (589 *&c.* the *Answer* required.

Quest. 5. A Bill of Exchange was accepted at *London* for the Payment of 400 *l.* *Sterling*, for the like Value delivered in *Amsterdam*, at 1 *l.* 13 *s.* 6 *d.* for 1 *l.* *Sterling*. How much Money was delivered at *Amsterdam*?
Answer, 670 *l.* *Flemish*.
For 1 *l.*=240 *d.* and 1 *l.* 13 *s.* 6 *d.*=402 *d.*
Then 240 : 402 : : 400 : 670 The *Answer* required.

Quest. 6. When the Exchange from *Antwerp* to *London* is at 1 *l.* 4 *s.* 7 *d.* *Flemish*, for 1 *l.* *Sterling*: How many Pounds *Sterling* must be paid at *London*; to ballance 236 *l.* *Flemish* at *Antwerp*?
Answer, 192 *l.* *Sterling*.
Thus 1 *l.* 4 *s.* 7 *d.*=295 *d.* and 1 *l.*=240 *d.*
Then 295 : 240 : : 236 : 192. The *Answer*.

Quest. 7. A Merchant delivered at *London* 120 l. *Sterling*, to receive 147 l. *Flemish* in *Amsterdam*. How much was 1 l. *Sterling* valued at, in *Flemish* Money?

Answer, 1 l. 4 s. 6 d.

Thus 120 : 147 :: 240 *d.* : 294 *d.*=1 l. 4 s. 6 *d.* &c.

Quest. 8. A Factor hath sold Goods at *Cadiz* for 1468 Pieces of Eight; valued at 4 s. 6½ *d. Sterling per* Piece. How much *Sterling* Money do those Pieces of Eight amount to.

Answer, 333 l. 7 s. 2 *d.*

Thus, if 1=54,5 *d.* then 1468×54,5=80006 *d.* &c.

Quest. 9. A Traveller would have an equal Number of Crowns at 5 s. 6 *d. per* Crown; and Dollers at 4 s. 5 *d. per* Piece; How many of each Sort may he have for 309 l. 8 s.

Answ. 624 of each.

Thus 309 l. 8 s. =74256 *d.*
And 5 s. 6 *d.*+4 s. 5 *d.*=119 *d.*
Then 119) 74256 (624. the *Answer* required.

Quest. 10. Suppose I would exchange 527 l. 17 s. 6 *d.* for Dollers at 4 s. 6 *d.* a Piece, Ducats at 5 s. 8 *d.* a Piece, and Crowns at 6 s. 1 *d.* a Piece. And would have 2 Dollers for 1 Ducat, and 3 Dollers for 2 Crowns. How many of each sort must I have?

Answ. 927 Dollers, 463½ Ducats, and 618 Crowns.

$$\text{For}\left\{\begin{array}{l}54\ d. = 1\ \text{Doller.}\\ 68\ d. = 1\ \text{Ducat.}\\ 73\ d. = 1\ \text{Crown.}\end{array}\right\}\ per\ \text{Question.}$$

And 126690 *d.*= 527 l. 17 s. 6 *d.*

Now if the Crowns, Dollers, and Ducats were to be equal in Number; then 73+54+68 must have been the Divisor, by which 126690 must have been divided, and the Quotient would have been the Answer to the Question. As in the last Example.

But here instead of their Sum, such Parts of them must be taken as are assigned or limited by the Question; that so the Number of some one of them may be found.

$$\text{And because there must be}\left\{\begin{array}{l}2\ \text{Dollers for 1 Ducat, and}\\ 3\ \text{Dollers for 2 Crowns.}\end{array}\right.$$

Therefore it will be but ½ of a Ducat for one Doller, and ⅔ of a Crown for one Doller.

Con-

Conſequently, $54 + \frac{68}{2} : + \frac{2}{3}$ of $73 = 136\frac{2}{3}$ Or $\frac{410}{3}$ will be the Diviſor to find the Number of Dollers.

Thus $\frac{410}{3}$) 126690 (927 the Number of Dollers.
Then $\frac{1}{2}$ of $927 = 463\frac{1}{2}$ is the Number of Ducats.
And $\frac{2}{3}$ of $927 = 618$ is the Number of Crowns.

Or if you pleaſe you may form Diviſors to find either the Ducats or Crowns firſt: For if it be 2 Dollers for 1 Ducat, and 3 Dollers for 2 Crowns, as before;
Then will 6 Dollers be for 3 Ducats, and 6 Dollers for 4 Crowns.

$$\text{Therefore,} \left\{ \begin{array}{l} \frac{3}{2} \text{ of a Doller} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\} \text{will be for 1 Crown.}$$

Conſequently, $\frac{2}{3}$ of $54 : + \frac{4}{3}$ of $68 : + 73 = 205$ will be the Diviſor to find the Crowns firſt, *&c.*

Queſt. 11. A Caſhier is to receive 500 *l.* He is offered Crowns at 6 *s.* $1\frac{1}{2}$ *d. per Crown*, which are worth but 6 *s.* Or he may have Dollers at 4 *s.* 5 *d.* the Piece, which are worth but 4 *s.* 4 *d.* Which of theſe ſhall he receive to have the leaſt Loſs? And how much will he loſe in the Payment?

$$1 \left\{ \begin{array}{l} 1 \text{ Crown} = 72\, d. \\ 1 \text{ Doller} = 52\, d. \end{array} \right\} \text{according to the true Values.}$$

$$2 \left\{ \begin{array}{l} 1 \text{ Crown} = 73{,}5\, d \\ 1 \text{ Doller} = 53\ \ d. \end{array} \right\} \text{the advanced Values.}$$

Now to find which will be the leaſt Loſs; find what the advanced Value of a Doller ought to be in Proportion to that of 1 Crown.
Thus 72 : 73.5 : : 52 : 53.22, &c. But he may have Dollers at 53 *d. per* Piece, therefore the Payment in Dollers will be the leaſt Loſs; *viz.* 53 is leſs than 53.22. &c.
Next, to find what the whole Loſs will be, divide 120000 *d.* = 500 *l.* by 52. and by 53. The Difference of their Quotients will be the Loſs.
Thus 52) 120000 ($2307\frac{9}{13}$. And 53) 120000 ($2264\frac{8}{53}$.
Then $2307\frac{9}{13} - 2264\frac{8}{53} = 43\frac{373}{689}$ Dollers at 4 *s.* 4 *d.* is the Loſs; *viz.* 9 *l.* 8 *s.* $10\frac{226}{689}$ *d.*

There are other Ways of anſwering the laſt Queſtion, but this I take to be the eaſieſt.

Queſt. 12. Suppoſe I exchange 4 *l.* 10 *s.* 10 *d.* for 11 Crowns and 7 Dollers; And at another time I have 4 Crowns and 3 Dollers

Dollers for 1 l. 15 s. each being of the ſame Value with the firſt. What is the Value of a Crown, and of a Doller?

First $\quad$ 11 Crowns + 7 Dollers = 1090 *d.* }
Second $\quad$ 4 Crowns + 3 Dollers = 420 *d.* } by the Queſtion.

Then in order to find the Value of 1 Crown, you muſt caſt off the Dollers by making them of the ſame Number; Thus,

33 Crowns + 21 Dollers = 3270 *d.* the firſt Multipl. with 3.
28 Crowns + 21 Dollers = 2940 *d.* the ſecond Multipl. with 7.

Then 5 Crowns = 330 *d.* being their Difference.
Conſequently 5) 330 (66 = 5 s. 6 *d.* is the Value of 1 Crown.
And 4 Crowns = 264 *d.*
Then will 3 Dollers = 420 *d.* — 264 *d.* = 156 *d.*
Conſequently 3) 156 (52 *d.* = 4 s. 4 *d.* the Value of 1 Doller.

CHAP. IX.

Of Alligation.

WHEN it is required to mix ſeveral Sorts of Ingredients together; as different Sorts of Corns, Wines, Wooll, Spices, or Metals; or to compoſe Medicines, &c. the Method of proportioning ſuch Mixtures, is called the *Rule* of *Alligation.* And is divided into two Parts or Branches; called MEDIAL and ALTERNATE.

Sect. 1. *Of* Alligation Medial.

Alligation Medial, is that by which the mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and their Rates are given: And is thus performed.

Firſt find the Sum of all the Quantities propoſed to be mix'd; And alſo the Sum of all their particular Rates.

Then the Proportion will be.

Rule. { *As the* Sum *of all the Quantities : Is to the* Sum *of all their Rates : : So is any Part of the Mixture : To the Mean Rate or Price of that Part.*

Queſt. 1. Suppoſe 15 Buſhels of Wheat at 5 s. the Buſhel, and 12 Buſhels of Rye at 3 s. 6 *d.* the Buſhel, were mix'd together; What

What is the Mean Rate or Price, it may be ſold for a Buſhel, without Loſs or Gain ;

This Queſtion prepared as directed above will ſtand

Thus { 15 Buſhels of Wheat at 5 *s. per* Buſhel, comes to 900 *d.*
{ 12 Buſhels of Rye at 3 *s.* 6 *d.* each, comes to 504 *d.*

27=their Sum. And their Total Value=1404 *d.*

Then 27 Buſhels : 1404 *d.* : : 1 Buſhel : 52 *d.*=4 *s.* 4 *d.* the *Anſwer* required.

Queſt. 2. A Grocer mixeth 36 ℔ of Tobacco, worth 1 *s.* 6 *d.* a Pound, with 12 ℔ of another Sort at 2 *s.* a ℔. And 12 ℔ of a third Sort at 1 *s.* 10 *d.* the Pound. How may he ſell the Mixture *per* Pound ?

	℔	*s.*	*d.*		
Firſt	36 . at	1 .	6	*per* ℔ Amounts to	648 *d.*
	12 . at	2 .	0		288 *d.*
	12 . at	1 .	10		264 *d.*

60 = the Number of ℔s. their Value = 1200 *d.*

Then 60 ℔ : 1200 *d.* : : 1 ℔ : 20 *d.* = 1 *s.* 8 *d.* the *Anſwer* required.

Queſt. 3. A Vintner mixeth $31\frac{1}{2}$ Gallons of *Malaga* Sack, worth 7 *s.* 6 *d.* the Gallon; with 18 Gallons of *Canary* at 6 *s.* 9 *d.* the Gallon; $13\frac{1}{2}$ Gallons of *Sherry* at 5 *s.* the Gallon ; And 27 Gallons of White Wine at 4 *s.* 3 *d.* the Gallon. 'Tis required to find what one Gallon of this Mixture is worth.

	Gal.	*s.*	*d.*		*Pence*
Firſt,	$31\frac{1}{2}$ at	7 .	6	*per* Gallon comes to	2835
	18 at	6 .	9		1458
	$13\frac{1}{2}$ at	5 .	0		810
	27 at	4 .	3		1377

90= the Number of Gallons. Their Value=6480

Then 90 : 6480 : : 1 : 72 *d.*=6 *s.* The Rate or Price of one Gallon, as was required.

The *Proof* of all Operations in theſe Sort of Mixtures, is done by comparing the Value of all the Mixture, being ſold at the Mean Rate; with the total Value of all the particular Quantities, ſuppoſing they had been ſold at their reſpective Rates unmix'd; if thoſe Sums are equal the Work is true.

Sect.

Sect. 2. *Of* Alligation Alternate.

Alligation Alternate, is that by which the particular Quantities of every Ingredient concern'd in any Mixture are found; when the particular Rates of every one of those Ingredients, and the Mean Rate are given; being (as it were) the Converse to *Alligation Medial*; as will appear by the following Operations, which admit of three *Cases*.

Case I. The particular Rates of any Ingredients proposed to be mixed, and the Mean Rate of the whole Mixture being given. To find how much of each Ingredient is requisite to compose the Mixture; when the whole Quantity, or any Part thereof is not limited.

Quest. 1. How much Wheat at 5 *s.* the Bushel, and Rye at 3 *s.* 6 *d.* the Bushel, will compose a Mixture that may be sold for 4 *s.* 4 *d.* the Bushel?

Note, In all Questions of this Nature, it will be convenient to place the Mean Rate so, as that it may be easily compared with the particular Rates, in order to find every one of their Differences from the Mean Rate, by Inspection only.

Thus, the Mean Rate = 52 *d.* $\begin{cases} \text{Wheat } 60\ d. \\ \text{Rye } 42\ d. \end{cases}$

Then take the several Differences between the Mean Rate, and the particular Rates; setting down those Differences alternately, and they will be the Quantities required.

$$\text{Thus } 52 \begin{cases} 60 \\ 42 \end{cases} \quad \begin{cases} 10 = 52 - 42 \\ 8 = 60 - 52 \end{cases}$$

That is 52—42=10 for the Quantity of Wheat.
And 60—52= 8 for the Quantity of Rye, that will compose the Mixture rquired.

The Proof by *Alligation Medial*.

Add	10 Bushels of Wheat at 60 *d. per* Bushel	= 600 *d.*
	8 Bushels of Rye at 42 *d. per* Bushel	= 336 *d.*
18 =	The Number of Bushels	= 936 *d.*

Then 18 : 936 :: 1 : 52 *d.* = 4 *s.* 4 *d.* the Mean Rate.

Note, Altho' 10 and 8 do answer the Question, as plainly appears by the Proof; yet they are not the only two Numbers; for this Question, and all others of this Kind, will admit of various Answers, and all in whole Numbers; for any two Numbers that are in the same Proportion to one another, as 10 is to 8, will as truly answer the Question.

Viz.

$$\textit{Viz.}\ 10 : 8 :: \left\{\begin{array}{c} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{array}\right\} \text{\&c. } \textit{ad infinitum.}$$

Quest. 2. A Grocer would mix three Sorts of Tobacco together, *viz.* One Sort of 18 *d. per* ℔. another Sort of 22 *d. per* ℔. and a third Sort of 2 *s.* the ℔. How much of each Sort must he take, that the whole Mixture may be sold for 20 *d.* the Pound.

Having set down the given Rates, as before: Then find each of their Differences from the proposed Mean Rate; and place those Differences Alternately. Thus,

$$\text{Mean Rate } 20 \left\{\begin{array}{c} 18 \\ 22 \\ 24 \end{array}\right\} \left\{\begin{array}{l} 4+2=24-20 \text{ and } 22-20 \\ 2=20-18 \\ 2=20-18 \end{array}\right.$$

These Differences, *viz.* 6 . 2 . 2 are the Quantities required.

$$\text{Proof} \left\{\begin{array}{l} 6 \text{ ℔. of Tobacco at } 18\ d.\ per \text{ ℔. comes to} \\ 2 \text{ ℔. at } 22\ d. \text{ the Pound comes to} \\ 2 \text{ ℔. at } 24\ d. \text{ the Pound comes to} \end{array}\right. \quad \left.\begin{array}{r} 108 \\ 44 \\ 48 \end{array}\right\} d.$$

10 = the Number of ℔s. Their Value = 200 *d.*

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the same *Ratio* to one another, as 6 . & 2 have, will answer the Question.

$$\text{That is } 6 : 2 :: \left\{\begin{array}{c} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{array}\right\} \text{\&c.}$$

But if only one of the three given Rates had been greater than the Mean Rate; as suppose 14 *d. per* ℔. 18 *d. per* ℔. and 24 *d. per* ℔. And the Mean Rate 20 *d.* as before. Then their Differences must have been placed,

$$\text{Thus } 20 \left\{\begin{array}{c} 14 \\ 18 \\ 24 \end{array}\right\} \left\{\begin{array}{c} 4 \\ 4 \\ 6+2 \end{array}\right\} \text{\&c. As before.}$$

Quest. 3. A Vintner would make a Mixture of *Malaga*, worth 7 *s.* 6 *d. per* Gallon, with *Canary* at 6 *s.* 9 *d. per* Gallon. *Sherry* at 5 *s. per* Gallon, and *White Wine* at 4 *s.* 3 *d. per* Gallon; What Quantity of each Sort must he take, that the Mixture may be sold for 6 *s.* a Gallon.

In all Questions of this kind, wherein it is required to mix four Things together, two of them having their Prices greater,

and two leſſer than the Mean Rate; you muſt always Alligate or Compare a greater and leſſer Price with the Mean Price, ſetting down their Differences Alternately, as in the firſt *Example* of this *Section*.

Thus, Mean Rate=72 *d.*	*Malaga*	90 *d.*	21=72—51	
	White	51 *d.*	18=90—72	
	Sherry	60 *d.*	9=81—72	
	Canary	81 *d.*	12=72—60	

Hence 21 Gallons of *Malaga*, 12 of *Canary*, 9 of *Sherry*, and 18 of White, will compoſe the Mixture required.

Or thus, 72	*Malaga*	90 *d.*	12 *Malaga*	will, *&c.*
	Sherry	60 *d.*	18 *Sherry*	
	Canary	81 *d.*	21 *Canary*	
	White	51 *d.*	9 *White.*	

Either of theſe Mixtures equally anſwer the Queſtion, which may be eaſily try'd as before in the laſt, *&c.*

Caſe II. The particular Rates of all the Ingredients propoſed to be mix'd, the Mean Rate of the whole Mixture, and any one of the Quantities to be mix'd being given. Thence to find how much of every one of the other Ingredients is requiſite to compoſe the Mixture.

Note, This is uſually called *Alligation Partial*.

Queſt. 4. How much Wheat at 5 *s.* the Buſhel, muſt be mix'd with 12 Buſhels of Rye at 3 *s.* 6 *d.* a Buſhel; that the whole Mixture may be ſold for 4 *s.* 4 *d.* the Buſhel.

In this Caſe you muſt ſet down all the particular Rates with the Mean Rate, and find their Differences juſt as before; without any Regard had to the Quantity given.

Thus, Mean Rate 52 *d.*	Wheat	60 *d.*	10
	Rye	42 *d.*	8

Then { *As the Quantity found by the Differences of the ſame* Name *with the Quantity given : Is to the Quantity given : : So is any of the other Quantities found by the Differences : To the Quantity of its Name.*

Thus 8 : 12 : : 10 : 15. the Quantity or Number of Buſhels of Wheat required.

Queſt. 5. How much *Malaga* at 7 *s.* 6 *d.* the Gallon, *Sherry* at 5 *s.* the Gallon, and *White Wine* at 4 *s.* 3 *d.* the Gallon, muſt be mix'd with 18 Gallons of *Canary* at 6 *s.* 9 *d.* the Gallon: That the whole Mixture may be ſold for 6 *s.* the Gallon.

The

The Terms being set down, *&c.* as before, will stand

$$\text{Thus, Mean Rate } 72\,d. \left\{\begin{array}{ll} \textit{Malaga} & 90\,d. \\ \textit{White} & 51\,d. \\ \textit{Sherry} & 60\,d. \\ \textit{Canary} & 81\,d. \end{array}\right. \quad \begin{array}{r} 21 \\ 18 \\ 9 \\ 12 \end{array}$$

$$\text{Then, As } 12 : 18 :: \left\{\begin{array}{l} 21 : 31\tfrac{1}{2} \text{ Gallons of } \textit{Malaga.} \\ 18 : 27 \text{ Gallons of } \textit{White.} \\ 9 : 13\tfrac{1}{2} \text{ Gallons of } \textit{Sherry.} \end{array}\right.$$

That is, $31\frac{1}{2}$ Gallons of *Malaga*, 27 of *White Wine*, and $13\frac{1}{2}$ of *Sherry*, being mix'd with 18 Gallons of *Canary*, will make the Mixture required.

$$\text{Or Thus, } 72 \left\{\begin{array}{ll} \textit{Malaga} & 90 \\ \textit{Sherry} & 60 \\ \textit{Canary} & 81 \\ \textit{White} & 51 \end{array}\right. \quad \begin{array}{r} 12 \\ 18 \\ 21 \\ 9 \end{array}$$

$$\text{Then, As } 21 : 18 :: \left\{\begin{array}{ll} 12 : 10\tfrac{6}{21} & \text{The } \textit{Malaga} \\ 18 : 15\tfrac{9}{21} & \text{The } \textit{Sherry} \\ 9 : 7\tfrac{15}{21} & \text{The } \textit{White Wine} \end{array}\right\} \text{\&c.}$$

$$\text{Proof} \left\{\begin{array}{lll} \textit{Gallons.} & & \textit{Pence.} \\ 10\tfrac{6}{21} & \text{at } 90\,d. \text{ each} = & 925\tfrac{15}{21} \\ 15\tfrac{9}{21} & \text{at } 60\,d. \text{ each} = & 925\tfrac{15}{21} \\ 7\tfrac{15}{21} & \text{at } 51\,d. \text{ each} = & 393\tfrac{9}{21} \\ 18 & \text{at } 81\,d. \text{ each} = & 1458 \\ \hline 51\tfrac{9}{21} & \text{Value} = & 3702\tfrac{18}{21} \end{array}\right.$$

Then $51\frac{9}{21}$) $3702\frac{18}{21}$ ($72\,d.$=$6\,s.$ the Mean Rate.

Therefore the Quantities are as truly assigned here, as in the last Work.

Case III. The particular Rates of all the Ingredients proposed to be mix'd; and the Sum of all their Quantities, with the Mean Rate of that Sum being given; To find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed.

Set down all the particular Rates with the Mean Rate, and find their Differences, as before: Add together all the Differences into one Sum;

Then { *As the* Sum *of all the Differences : Is to the* Sum *of all the Quantities given :: So is every particular Difference : To its particular Quantity.*

Quest. 6. Let it be required to mix Wheat at 5 *s.* the Bushel, with Rye at 3 *s.* 6 *d.* the Bushel; So as that the whole Quantity may be 27 Bushels, to be sold for 4 *s.* 4 *d.* a Bushel. What Quantity of each must be taken to make up the Mixture?

Mean Rate 52 { Wheat 60 *d.* } { 10
Rye 42 *d.* } { 8

18=their Sum.

Then 18 : 27 : : { 10 : 15 } { The Quantities required.
8 : 12 }

Queſtion 7. Suppoſe it were required to mix *Malaga* at 7 *s.* 6 *d.* the Gallon, with *Canary* at 6 *s.* 9 *d.* the Gallon; *Sherry* at 5 *s.* the Gallon; and *White Wine* at 4 *s.* 3 *d.* the Gallon: So as that the whole Mixture may be 90 Gallons; to be ſold for 6 *s.* the Gallon. How much of each Sort will compoſe that Mixture ?

Mean Rate=72 *d.*	*Malaga*	90	21
	White	51	18
	Canary	81	9
	Sherry	60	12

60=their Sum

Then { 60 : 90 : : 21 : $31\frac{1}{2}$ The Gallons of *Malaga.*
60 : 90 : : 18 : 27 The Gallons of *White Wine.*
60 : 90 : : 9 : $13\frac{1}{2}$ The Gallons of *Canary.*
60 : 90 : : 12 : 18 The Gallons of *Sherry.*

Or thus 72	*Malaga*	90	12
	Sherry	60	18
	Canary	81	21
	White	51	9

60 their Sum

Then { 60 : 90 : 12 : 18 Gallons of *Malaga*
60 : 90 : 18 : 27 Gallons of *Sherry*
60 : 90 : 21 : $31\frac{1}{2}$ Gallons of *Canary*
60 : 90 : 9 : $13\frac{1}{2}$ Gallons of *White Wine.*

Either of theſe Ways do equally anſwer the Queſtion, as may be eaſily tried by *Alligation Medial.* As before, *&c.*

Note, The Work of theſe Proportions may be much ſhortned (eſpecially when there are many Ingredients to be mix'd) if you obſerve the ſame Method as was propoſed in the *Rule of Fellowſhip*, Page 99, *&c.*

I have made uſe of the very ſame *Examples* both in *Alligation Medial* and *Alternate*, throughout the three *Caſes*; being, as I preſume, much better than if they had been different ones; becauſe the Learner may (if he conſider a little on them) eaſily perceive, not only the Difference between the two Rules, but alſo wherein

in the chief Difference of each Case in the *Alternate Rule* depends, *&c.* Not but that I could have inserted many various *Examples*, as also the Manner of composing Medicines, *&c.* which, for Brevity's sake, I have omitted, and refer those that desire to see into that Business to Sir *Jonas More*'s Arithmetick, wherein he will find it largely handled. And so I shall conclude with *Alligation Alternate*, which altho' it gives true Answers to Questions of that kind, with some little Variety, according as the Ingredients are more or less in Number; as appears by the foregoing Examples. Yet it will not give all the Answers as such Questions are capable of, nor perhaps those which suit best with the present Occasion: Nor can this Imperfection be remedied by common Arithmetick; but by an Algebraic Way of Arguing it may; whereby all the possible Answers to any Question may be clearly and easily discovered; As shall be shew'd further on in the second Part.

CHAP X.

Of Metals, *and their* Specific Gravities, &c.

Sect. 1. *Of* Gold *and* Silver.

PURE GOLD, free from Mixture of other Metals, usually called *Fine Gold*, is of such a Nature and Purity that it will endure the Fire without Wasting; although it were kept continually melted: And therefore some of the *Ancient Philosophers* have supposed the *Sun* to be a *Globe* of *Liquid* or *Melted Gold*.

Silver having not the Purity of *Gold*, will not endure the Fire like it; yet *Fine Silver* will wast but a very little by being in the Fire any reasonable Time; whereas *Copper*, *Tin* *Lead*, *&c.* will not only wast, but may be calcin'd or burnt to a Powder.

Both *Gold* and *Silver* in their Purity, are so very flexible or soft (like new Lead, *&c.*) that they are not so useful either in Coin, or otherwise (except to beat into *Leaf-Gold* or *Silver*) as when they are allay'd, or mix'd and harden'd with *Copper* or *Brass*. And altho' most Places differ more or less in the Quantity of such Allay, yet in *England* it is generally agreed on, that,

Standard

Standard for Gold.

22 Carracts of Fine Gold, and 2 Carracts of Copper, being melted together, ſhall be eſteemed the true Standard for Gold Coin, *&c.* The *French* and *Spaniſh* Gold being very near of the ſame Standard.

That is, if any Quantity or Weight of *Fine Gold*, be divided into twenty four equal Parts, and 22 of thoſe Parts be mix'd with 2 of the like Parts of Copper; that Mixture is called *Standard Gold.*

Whence you may obſerve, that a Carract is not any certain Quantity or Weight, but $\frac{1}{24}$ Part of any Quantity or Weight; and the *Minters* and *Goldſmiths* divide it into 4 equal Parts, which they call Grains of a Carract; alſo they ſubdivide one of thoſe Grains into Halves, Quarters, *&c.*

Standard for Silver.

Eleven Ounces and Two-penny Weight of Fine Silver, and Eighteen Penny-weight of Copper being melted together, is eſteem'd the true Standard for Silver Coin; called *Sterling Silver.* And ſo in Proportion for a greater or leſſer Quantity; which is a leſs Proportion of Allay for Silver, than the other is for Gold.

Note, When either Silver or Gold, is finer than Standard, 'tis called *Better*; if courſer, 'tis called *Worſe*; and that Betterneſs or Worſeneſs, is reckoned by Carracts and Grains of a Carract in Gold; and by Penny-weights in Silver, and is thus diſcovered: The Goldſmiths, or Refiners, *&c.* do take a ſmall Quantity of ſuch Gold as they intend to try (which they call making an Aſſay) and weigh it very exactly, then they put it into a *Crucible*, and melt it in a ſtrong Fire, ſo long that if there be any Copper, or other Allay mixt with it, that Allay may be conſum'd or burnt away: When 'tis cold they weigh it very exactly again, and if it have loſt nothing of its firſt Weight, they conclude it is Fine Gold, but if the Loſs be $\frac{1}{24}$ Part, they call it 23 Carracts Fine, or one Carract better than Standard: If it have loſt $\frac{2}{24}$ Parts, 'tis 22 Caracts fine, or Standard: If $\frac{3}{24}$ Parts, it is ſaid to be 21 Carracts fine, or rather one Carract worſe than Standard, and ſo in Proportion as it happens to be *Better*, or *Worſe*.

In the ſame manner they make their Aſſay on Silver, only they compute its Loſs by Penny-weights, *&c.*

The Author of the *Preſent State of* England, mention'd before (*Page* 32) ſays,

'That

'That the *English* Coin may not want neither the Purity nor
'Weight required, it is most wisely and carefully provided, that
'once every Year the chief Officers of the *Mint* appear before
'the Lords of the Council in the *Star-Chamber* at *Westminster*, with
'some Pieces of all Sorts of Moneys coined the foregoing Year,
'taken at Adventure out of the *Mint*, and kept under several
'Locks, by several Persons, till that Appearance, and then by a
'Jury of 24 able Goldsmiths, in the Presence of the said Lords,
'every Piece is most exactly weighed and assay'd.

This if it were constantly practised would keep our Coin to its true Standard, *&c.*

Many pretty Questions may be started concerning the Fineness of Gold and Silver, *&c.*

Example 1.

If an Ingot of Silver weighing 787 *oz.* 14 *pwt.* 6 *grains*, be 11 *oz.* 6 *pwt.* fine: How much fine Silver is there in it, and what amounts it to, at 5 *s.* $1\frac{1}{2}$ *d.* the Ounce?

This Ingot is better than Standard by 4 *pwt.* For 11 *oz.* 2 *pwt.* = 222 *pwt.* the fine Silver in 12 *oz.* of Standard. But 11 *oz.* 6 *pwt.* = 226 *pwt.* the fine Silver in 12 *oz.* according to the Question.

First 787 *oz.* 14 *pwt.* 6 *gr.* = 378102 *Grains.*
And 12 *oz.* = 240 *pwt.*

Then as 240 : 226 : : 378102 : $356046\frac{1}{20}$ = 741 *oz.* 15 *pwt.* $6\frac{1}{20}$ *gr.* the fine Silver in that Ingot.

Which at 5 *s.* $1\frac{1}{2}$ *d.* the Ounce, amounts to 190 *l.* 1 *s.* 6 *d.* and near a *half-penny.*

Example 2.

If an Ingot of Gold weighing 115 *oz.* 13 *pwt.* 18 *grains*; Be $\frac{1}{4}$ of a *grain* worse than Standard: How much Standard Gold is there in it, and what comes it to at 3 *l.* 11 *s.* an Ounce?

First 115 *oz.* 13 *pwt.* 18 *gr.* = 55530 *Grains Troy.*
Then 24) 55530 (2313,75 = a Carract of that Quantity.
And 4) 2313,75 (578,4375 = a Grain of that Carract.
Consequently 4) 578,4375 (144,609375 = $\frac{1}{4}$ of a Grain.

Again, 2313,75×22 = 50902,5 ought to be the fine Gold in that Ingot, if it had been Standard:

But

But $50902,5 - 144.609375 = 50757,890625$ is the Quantity of fine Gold according to the Question.

Therefore $50925 : 50757,890625 :: 55530 : 55372.244$ &c. Grains = 115 *oz.* 7 *pwt.* 4.244 &c. Grains *Troy*, being the Quantity of Standard Gold in that Ingot. As was required.

Next for the Value of it, at 3 *l.* 11 *s. per* Ounce; 1 *oz.* = 480 Grains. And 3 *l.* 11 *s.* = 71 *s.*

Consequently $480 : 71 :: 55372.244$ &c. : 8190.4777 &c. = 409 *l.* 10 *s.* $5\frac{3}{4}$ *d.* very near, being the Value of that Ingot. As was required.

Or the last Question may be otherwise wrought thus; 115 *oz.* 13 *pwt.* 18 *grains* = 115,6875. And $\frac{1}{4}$ of a Grain, of a Carract is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$)

Then $22 - \frac{1}{16} = 21\frac{15}{16} = 21,9375$.

Consequently $22 : 21,9375 :: 115.6875 : 115,358842$ &c. = 115 *oz.* 7 *pwt.* 4.244 Grains, &c. As before.

Next for the Value, As $1 : 3,55 :: 115,358842 : 409,523889$ = 409 *l.* 10 *s.* $5\frac{3}{4}$ *d.* very near. As before.

Sect 2. *The* Specific Gravity *of* Metals, &c.

I take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curiosity, but also) of very good Use upon many Occasions. Therefore several Authors have given us such Proportions, or Difference of their Weights, as they are said to have one to another; supposing every one of them to be of the same Magnitude or Bigness. Some of which I shall here insert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, Printed *Anno* 1633, sets down the Proportions of their Weights. Thus;

Gold 1875. Lead 1165. Silver 1040. Copper 910. Iron 810. Tin 750. Water 100.

2. One *Alsted*, in his *Encyclopædia*, Printed 1649, hath them Thus;

Gold 1875. Quickfilver 1500. Lead 1165. Silver 1040. Copper 910. Iron 806. Tin 750. Honey 150. Water 100. Oil 90. These seem to be taken from those of *Van Etten*'s, with some *Additions* only.

3. The Ingenious Mr. *Oughtred*, in his *Circles of Proportions*, Printed *Anno* 1660. hath their Proportions according to the *Experiments* of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*. Thus;

Gold 3990. Quickfilver 2850. Lead 2415. Silver 2170. Brass 1890. Iron 1680. Tin 1554.

4. In

4. In the *Philosophical Transactions*, (Number 169 and 199) there is an Account of a great many Experiments of this kind; from whence I collected these following. *Viz.* Gold 18888. Mercury 14019. Lead 11343. Silver 11087. Copper 8843. Hammer'd Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of, and published by Order of the Royal Society, seem to be unquestionably true: Nevertheless because they differ so much from the before-mentioned (*and those from one another*) I have for my own Satisfaction made several Experiments of that kind: And have (*I presume*) obtained the Proportions of Weight that one Body bears to another, of the same Bulk or Magnitude, as nicely as the Nature of such Matter, as may be contracted or brought into a lesser Body (*viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as followeth.

		Ounces Troy.		*Ounces Aver.*
A *Cubic Inch* of	Fine Gold is	10,359273	=	11,365602
	Standard Gold,	9,962625	=	10,930422
	Quickſilver,	7,384411	=	8,101753
	Lead,	5,984010	=	6,553885
	Fine-Silver,	5,850035	=	6,418324
	Standard Silver,	5,556769	=	6,096569
	Roſe-Copper,	4,747121	=	5,208369
	Plate-Braſs,	4,404273	=	4,832116
	Caſt-Braſs,	4,272409	=	4,630300
	Steel,	4,142127	=	4,544505
	Common Iron,	4,031361	=	4,422979
	Block-Tin,	3,861519	=	4,236638
	Fine Marble,	1,429411	=	1,568859
	Common Glaſs,	1,360841	=	1,493037
	Alabaſter,	0,988456	=	1,084477
	Dry Ivory,	0,962083	=	1,055542
	Dry Box wood,	0,543282	=	0,596057
	Sea-Water.	0,542742	=	0,594894
	Common Clear Water,	0,527458	=	0,578697
	Red Wine,	0,523766	=	0,574646
	Proof Spirits or Brandy,	0,489268	=	0,536796
	Sound dry Oak	0,489008	=	0,536569
	Lin-ſeed Oil,	0,491591	=	0,539345
	Oil-Olive,	0,481569	=	0,528350

In this Table you have the Specific Gravity or Weight of a Cubic Inch, of various Sorts of Bodies, both in Troy Ounces and Averdupois Ounces, and Decimal Parts of an Ounce, which I can aſſure you required more Charge, Care and Trouble, to find out nicely, than I was at firſt aware of.

Now from hence it will be eaſy to determine the Weight of any propoſed Quantity, of the ſame Matter and Kind with thoſe in the Table; its ſolid Content being given in Cubic Inches. For it is plain, that if the Number of Cubic Inches contained in any given Quantity, be multiplied with the Tabular Weight of one Inch (of the ſame kind of Matter) the Product will be the Weight of that Quantity in Ounces, &c.

Example.

Suppoſe it were required to find the Weight of a Piece of Marble, containing three Solid Feet, and 40 Cubic Inches.

Firſt $1728 \times 3 = 5184$ the Cubic Inches in 3 Solid Feet.

And $5184 + 40 = 5224$ the Number of Cubic Inches in the Piece of Marble.

Then $5224 \times 1,429411 = 7410,066624$ Ounces Troy.
Or $5224 \times 1,568859 = 8195,719416$ Ounces Averdupois.

The Weight of that Piece of Marble, in Ounces, &c. which is eaſily brought into Pounds, &c. The like for any of the reſt.

The Converſe of this Work is eaſy; *viz.* If the Weight of any propoſed Quantity be given, thence to find the ſolid Content of that Quantity in Cubic Inches, &c.

Thus, *Divide the given Weight of the propoſed Quantity (it being firſt reduced into Ounces,* &c. *) by the Tabular Weight of one Inch (of the ſame kind of Matter) and the Quotient will be the Number of Cubic Inches contained in that Quantity.*

Note, If you would find what Weight any Quantity of thoſe Bodies mentioned in the Table will have, when it is immerſed or put into Water, you muſt ſubtract the Weight of an equal Quantity of Water (with that of the Body) from the Weight of the propoſed Body (if it be heavier than Water) and there will remain the Weight required. As for Inſtance,

$$\left.\begin{array}{l}\text{A Cubic Inch of Lead} = 5.984010 \\ \text{A Cubic Inch of Water} = 0,542742\end{array}\right\}\text{Ounces Troy, \&c.}$$

their Difference is, $= 5.441268$ the Weight of a Cubic Inch of Lead in the Water, &c.

CHAP.

CHAP. XI.

Evolution, *or Extracting the* Roots *out of all* Single Powers; *by one General Method.*

Sect. I.

E*Volution* is the *Unravelling*, or as it were the *Unfolding* and *Resolving* any proposed Power or Number, into the same Parts of which it was composed, or supposed to be made up. Now in order to perform that, it will be convenient to consider how those Powers are composed, *&c.*

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid* 7. *Def.* 18.

Thus the Square Number 4 is composed of the two equal Numbers 2 and 2. *viz.* $2\times2=4$.

Or the Square Number 9 is composed of the two equal Numbers 3 and 3. *viz.* $3\times3=9$. According to *Euclid.*

That is, if any Number be multiplied into it self; that Product is called a Square Number.

A Cube is that Number, which is equally equally equal, or which is contained under three equal Numbers. *Eucl.* 7. *Defi.* 19.

Thus the Cube Number 8 is composed of the three equal Numbers 2 and 2 and 2. *viz.* $2\times2\times2=8$, *&c.*

That is, if any Number be multiplied into it self, and that Product be multiplied with the same Number; the second Product is called a Cube Number.

These Two, *viz.* the Square, and Cube Numbers borrow their Names from Geometrical Extensions or Figures; as from the three signal Quantities mention'd in *Page* 2.

That is, a *Root* is represented by a *Line* or *Side*, having but one Dimension. *viz.* that of *Length* only.

The Square is a Plain or Figure of two Dimensions, having equal *Length* and *Breadth*. The Cube is a solid Body of three Dimensions; having equal *Length*. *Breadth*, and *Thickness*: But beyond these three, Nature proceeds not, as to local Extension. That is, the Nature of Place or Space, admits no Room for other Ways of Extension, than Length, Breadth and Thickness. Neither is it possible to form, or compose any Figure or Body beyond that of a *Solid.*

And therefore all the Superior Powers above the Cube or third Power; as the Biquadrat or fourth Power, the Sursolid or fifth Power, *&c.* are best explain'd and understood by a Rank or Series of Numbers in *Geometrical Proportion.*

For Instance:

Suppose any Rank of *Geometrical Proportionals*, whose first Term and Ratio are the same; And to them let there be assigned

a Series

a Series of Numbers in *Arithmetical Progression*, beginning with an Unit or 1. whose common Difference is also 1. as in *Page* 79.

Thus { 1 . 2 . 3 . 4 . 5 . 6 . 7 *Indices.*
2 . 4 . 8 . 16 . 32 . 64 . 128 *&c.* in ÷

Then are those Numbers in ÷ produced by a continued Multiplication of the first Term or Root into it self; And those in *Arithmetical Progression* or *Indices*, do shew what Degree or Power each Term in the *Geometrical Proportion* is of.

For *Example*; In this Series of ÷ 2 is both the first Term or Root, and common Ratio of the Series.

Then $2 \times 2 = 4$ the second Term or Square.

And $2 \times 2 \times 2 = 8$ Or $4 \times 2 = 8$ the Cube or third Term.

Again $2 \times 2 \times 2 \times 2 = 16$ Or $8 \times 2 = 16$ the fourth Term or Biquadrat. And so on for the rest.

Note, This is called *Involution*, viz. When any Number is drawn into it self, and afterwards into that Product, *&c.* 'tis said to be so often involved into it self; And the *Indices* are the Exponents of their respective Powers so involved.

And according to these Involutions, is formed the following *Table of Powers*; wherein the Root is only one single Figure.

Root, or single Side.	*Square or 2d Power.* Index (2)	*Cube, or the 3d Power.* Index (3)	*Biquadrat, or Square Squared; being the 4th Power.* Index (4)	*Sursolid, or the 5th Power.* Index (5)	*Square Cubed, or Cube Squared; the sixth Power.* Index (6)	*The second Sursolid, or seventh Power.* Index (7)	*The Biquadrat Squared, or the 8th Power.* Index (8)	*The Cube Cubed, or the ninth Power, &c.* Index (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This *Table* plainly shews (by Inspection) any Power (under the tenth) of all the nine Figures; and from thence may be taken

taken the neareſt Root of any Square, Cube, Biquadrat, *&c.* of any Number whoſe Root or Side is a ſingle Figure.

But if the Root conſiſt of two, three, or more Places of Figures, then it muſt be found by Piece-meal, or Figure after Figure, at ſeveral Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Surſolid, *&c.*) hath heretofore been a very tedious and troubleſom Piece of Work: All which is now very much ſhortned, and rendred eaſy, as will appear further on.

When any Number is propoſed to have its Root extracted, the firſt Work is to prepare it by Points ſet over (or under) their proper Figures; according as the given Power, whoſe Root is ſought, doth require; and that's done by conſidering the Index of the given Power, which for the Square is 2. for the Cube is 3. for the Biquadrat is 4. *&c.* (as in the preceding *Table*). Then allow ſo many Places of Figures in the given Power, for each ſingle Figure of the Root, as its Index denotes; always beginning thoſe Points over the Place of Unity, and aſcend toward the Left Hand if the given Number be Integers, and deſcend towards the Right Hand in Decimal Parts. As in theſe following.

Suppoſe any given Number; As *75640387246* which I all along hereafter call the Reſolvend.

Then if it be required to extract any of the following Roots, it muſt be pointed (according to the forementioned Conſideration) in this Manner.

Viz. For the	Square Root Thus	75640387246
	Cube Root	75640387246
	Biquadrat Root	75640387246
	Surſolid Root	75640387246

Or ſuppoſe the Number to be 0,674035982

Then for the	Square Root Thus	0,6740359820
	Cube Root	0,674035982
	Biquadrat Root	0,674035982000

Now the Reaſon of Pointing the given Reſolvend in this Manner; *viz.* the allowing two Figures in the Square; three Figures

Figures in the Cube, and four Figures in the Biquadrat, &c. For one Figure in the Root, may be made evident ſeveral Ways; but I think 'tis eaſily conceiv'd from the *Table of ſingle Powers*, wherein you may obſerve that all the Powers of the Figure 9, (which is but a ſingle Figure) have the ſame Number of Places of Figures, as the Index of thoſe Powers denotes: Therefore ſo many Places of Figures muſt be taken or aſſigned for every ſingle Figure in the Root. Conſequently by theſe Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, ſo many Figures there muſt be in the Root, and whether they muſt be Integers, or Decimal Parts, is eaſily determined by the reſpective Places of the Points.

Sect. 2. *To* Extract *the* Square Root.

And firſt how to extract the Square Root, according to the common Method.

Having pointed the given Reſolvend into Periods of two Figures, as before directed; then by the *Table of Powers*, (or otherwiſe) find the greateſt Square that is contained in the firſt Period towards the Left-Hand; (ſetting down its Root, like a Quotient Figure in Diviſion) and ſubtract that Square out of the ſaid Period of the Reſolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the firſt Square for a Diviſor; enquiring how oft it may be had in that Dividend; So as when the Quotient Figure is annexed to the Diviſor, and that increaſed Diviſor being multiplied with the ſame Quotient Figure, the Product may be the greateſt Number that can be taken out of that Dividend; which ſubtract from the ſaid Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then ſee how often the laſt increaſed Diviſor, can be had in the new Dividend; (with the ſame Caution as before, *viz.*) ſo as that the Quotient Figure being annexed to the Diviſor, and that increaſed Diviſor multiplied with the ſame Quotient Figure, their Product may be the greateſt Number that can be ſubtracted from the new Dividend. (As before). And ſo proceed on from Period to Period; (*viz.* from Point to Point) in the very ſame Manner, until all be finiſhed.

An *Example* or two being well obſerved will render the Work of forming the new Diviſors, &c. more plain and eaſy, than can be expreſſed in a Multitude of Words.

Example

Example 1. Let it be required to extract the Square Root out of 572199960721. This Resolvend being prepared or pointed as before directed, will stand

Thus 572199960721 (756439 the Root.
49 = the greatest Square in 57.

1 Divisor 145) 821
5 725 = 145 × 5
2 Divisor 1506) 9699
6 9036 = 1506 × 6
3 Divisor 15124) 66396
4 60496 × 15124 × 4
4 Divisor 151283) 590007
3 453849 = 151283 × 3
5 Divisor 1512869) 13615821
9 13615821 = 1512869 × 9

Proof 756439 × 756439 = 572199960721 the Resolvend.

Example 2. What's the Square Root of 1850701,764025?

Operation 1850701,764025 (1360,405
1
23) 85
3 69
266) 1607
6 1596
27204) 1101,76
4 1088,16
2720805) 13,604025
5 13,604025
(0)

Hence 1360,405 is the Root required.

Example 3. What's the Square Root of 0,06076225 Decimal Parts?

Operation 0,06076225 (0,2465 the Root required.
,04 = ,2 × 2
,44 207
4 176
,486 3162
6 2916
,4925) 24625
5 24625
(0)

Proof: 0,2465 × 0,2465 = 0,06076225 the Resolvend.

What

What is here done in Whole Numbers, Mix'd Numbers and Decimals, may alſo be done in Vulgar Fractions; if you firſt change the given Fraction into Decimals. (As in *Sect.* 5. *p.* 68.)

Example 4. Let it be required to extract the Square Root of $\frac{16}{25}$. Firſt $\frac{16}{25}=0{,}64$.

$$\begin{array}{l} \text{Then } 0{,}\dot{6}\dot{4} \;(\,{,}8 \text{ the Root required.} \\ \underline{\quad{,}64\quad} \\ \quad(0) \end{array}$$

In theſe four Examples the Reſolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Reſolvend is not a true figurate Number, according to the propoſed Power. That is, 'tis not a perfect Square, Cube, Biquadrat, *&c.* And then ſomething will remain after the Extraction hath been made throughout all the Points. Such Numbers are called *Surd* Numbers, and their Roots can never be truly found, but will become a continued Series *ad infinitum*: If to the Remainder there be ſtill annexed Cyphers according as the propoſed Power requires, *viz.* by Two's in the Square; Three's in the Cube; Four's in the Biquadrat, *&c.* And the Operations continued on as before.

Example 5. Suppoſe it were required to extract the Square Root of 6968.

Operation	
	6968 (83,4745, *&c.*
	64
163)	568
3	489
1664)	79,00
4	66 56
16687)	12 4400
7	11 6809
166944)	759100
4	667776
1669485)	9132400
5	8347425
1669490	784975 *&c.*

Thus the Root of any Surd Number may be continued on to what Exactneſs you pleaſe, but cannot be truly found.

In my *Compendium* of *Algebra*, Chap. 9. I have propoſed another Way of extracting the Square Root, and there given *Examples* of the Work: which to avoid Prolixity is thus;

Having

Having pointed the given Refolvend, and taken the greateſt Square to the firſt Point from it, as before. Then divide the Remainder of the whole Reſolvend by 2 (that is, halve it) and point it a new. (This I call a new Dividend.) Then make the Root of the firſt Square a Diviſor, enquiring how oft it may be found in the new Dividend to the next Figure forward, reſerving that Figure under the next Point, for the half Square of the Quotient Figure. Which being found, multiply the Diviſor with it, adding to that Product the Tens of the Half Square if there be any; As in plain Diviſion.

Then annex the Quotient Figure to the laſt Diviſor for a new Diviſor, with which proceed in all Reſpects as with the laſt Diviſor; and ſo until all be finiſhed.

Example 6. What's the Square Root of 2990667969

```
                • • • • •
Operation     2990667969
      —       25                ( 5 The firſt Single Root.
              ──────────
      2)       490667969   The Remainder to be divided by 2.
                • • • •
Firſt Root 5)  245333984,5  ( 54687
       + 4     208=5×4 : + ½ the Square of 4 viz. 16/2=8.
Diviſor  54)   3733
       + 6)    3258=54×6 : + ½  the Square of 6.
Diviſor 546)    47539
       + 8)     43712=546×8 : + ½ the Square of 8
Diviſor 5468)    382784,5
       + 7       382784,5=5468×7 : + ½ the Square of 7
                   (0)
```

Hence the Root is found to be 54687 As was required.

All the Difficulty in this Method is only in the true placing of the half Square of the Quotient Figure, when it happens to be an odd Number; In that Caſe you muſt bring down one Figure more of the Dividend; *viz.* of the next Period; under which, place the odd 5 that will always ariſe from the half Square of an odd Number: As 7 whoſe Square is 49; the half of which is 24,5 to be placed as in the laſt Operation of this Example.

N. B. When the Number of Figures in the Root of any Surd Number are limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the deſigned Number of Figures; for the reſt may be obtained by plain Diviſion only.

Example 7. Suppose it were required to extract the Square Root of 7 (a Surd Number) to have 12 Places of Figures in it.

	7	(2,645751 First Part of the Root.
	4	
Remainder	3	
2)	1,50	= half the Remainder.
+ ,6	1,38	= 2 × ,6 : + $\frac{1}{2}$ the Square of 0,6 = 0,18
2,6)	1200	
+ 0,4	1048	
2,64)	152000	
+ ,005	132125	
2,645)	1987500	
+ ,0007	1851745	
2,6457)	13575500	
+ ,00005	13228625	
2,64575)	34687500	
+ ,000001	26457505	
2,645751	8229995	

Having thus got 7 of the 12 Figures required in the Root; the rest may be easily found by the contracted Way of Division proposed in *Page* 68.

Thus, 2,645751)	8229995	(2,64575131106
....	7937253	
	292742	
	264575	
	28167	
	26457	
	1710	
	1697	
	(13)	

Hence I find the Root of 7 to be 2,6457131106 As was required.

Thus you have two Ways of extracting the Square Root, either of them may be practised as every one likes best.

Sect.

Sect. *To* Extract *the* Cube Root.

The Method that I ſhall here propoſe for extracting the Cube Root admits of two Caſes; both which are to be very well obſerved.

Having pointed the given Reſolvend, (as before directed) *viz.* into Periods of three Figures; then ſeek a Cube Number by the *Table* of *Powers* (or otherwiſe) that comes the neareſt to the firſt Period of the Reſolvend, whether it be greater or leſs than that Period.

Caſe I. If the Cube Number ſo taken, be leſs than the firſt Period of the Reſolvend.

Call its Root Leſs *than* Juſt.

And ſubtract that Cube from the firſt Period of the Reſolvend.

Caſe 2. But if that Cube be greater than the firſt Period of the Reſolvend.

Call its Root More *than* Juſt.

And ſubtract the Reſolvend from that Cube, annexing Cyphers to it, that ſo Subtraction may be made.

To the firſt Root whether it be leſs, or more than juſt, annex ſo many Cyphers as there are remaining Points over the whole Numbers of the Reſolvend, and multiply it with 3; then make that Product a Diviſor; by which you muſt divide the Difference between the Reſolvend and the foreſaid Cube, then will that Quotient be the Reſolvend depreſſed to a Square; and therefore it muſt be pointed as ſuch: *viz.* into Periods of two Figures each. That being done, make the firſt Root (without thoſe Cyphers that were annex'd to it) a Diviſor, enquiring how oft it may be found in the firſt Period of the new Reſolvend, (as before in extracting the Square Root) with this Conſideration, that if the Root, (now a Diviſor) be leſs than juſt, as in *Caſe* I. you muſt annex the Quotient Figure to it, and then multiply the Root ſo encreaſed, into the ſaid Quotient Figure; ſetting down the Units Place of their Product under the pointed Figure of that Period, ſubtracting it, as in Diviſion. And ſo on from one Period to another. As before.

But if the ſaid Root (now a Diviſor) be more than juſt, as in *Caſe* 2. Then you muſt ſubtract the Quotient Figure from a Cypher annexed, or ſuppoſed to be annexed to the ſaid Diviſor; multiplying the Root ſo decreaſed into the Quotient Figure; ſetting down their Product as before, *&c.* An Example or two in each Caſe will render the Work plain and eaſy.

S 2 *Example*

Example 1.

What's the Cube Root of 146363183 the given Resolvend, to be pointed thus 146363183 (5 the first Root, less than just.

125= the nearest Cube to 146

500×3=1500) 21363183 (14242,12 New Resolvend.

First Root 5) 14242,12 (527 the Root required.
+ 2 104

1 Divisor 52) 3842
+ 7 3689

2 Divisor 527 (153) the Remainder to be rejected.

Here the Root 527 is the true Root at the first Operation, as may be easily tried by involving it.

That is 527×527×527=146363183 the given Resolvend. But if it had not been the true Root; then every thing that hath been here done must have been repeated; only instead of the first single Root (*viz.* 5) you must have taken the encreased Root (*viz.* 527) and this I call a second Operation; which would increase the last Root to nine Places of Figures; *viz.* every Operation triples the Number of Places in the last Root; as will appear further on.

N. B It often happens that four, or sometimes five Places of Figures may be taken into the Root; especially when the second Place proves to be a Cypher. That is, when the first Cube comes very near to the first Period of the Resolvend.

Example 2.

What's the Cube Root of 67507824239 (4000 Root less than just,
First nearest Cube = 64

Root 4000×3=12000) 3507824239 (292318,68
4)
+ 0 292318,68 (4071,8

1 Divisor 40) 2923
+ 7 2849

2 Divisor 407 7418
+ 1 4071

3 Divisor 4071) 3347,68
+ 8 3257,44 &c.

Root = 4071 8

In this Example I have taken five Figures into the Root, because the second Place proved to be a Cypher. And in these five

five the Excess is not an Unit in the last Place; for if there were made a second Operation, the Root would be 4071,78 *&c.* as may be easily tried.

Example 3.

Let it be required to extract the Cube Root out of this Number;

Viz. 976379602989073960279630298890

The nearest Cube to 976 is 1000 whose Root is 10 being more than just.

Its Cube is 1000000000000000000000000000000

+ 976379602989073960279630298890 the Resolvend.

Remains 23620397010926039720369701110

The First Root 10000000000 × 3 = 30000000000 the Divisor

Then 30000000000) 23620397010926039720369701110 (*

0000000000

First Root	10)	*787346567030867990	New Resolvend.
	— 00	(007929	
1. Divisor	100	7873	
	— 7	6951	
2. Divisor	993	92246	first Root = 10000000000
	— 9	89289	— 007929
3. Divisor	9921	295756	9920710000
	— 2	198416	
4. Divisor	99208	9734070	
	— 9	8928639	
5 Divisor	992071	*&c.*	

At this first Operation I take but 99207 to which I annex 5 Cyphers for the remaining Points, *viz.* 9920700000 which being involved to the third Power or Cube, for a second Operation,

will be 976398156022743000000000000000

— 976379602989073960279630298890 Resolvend

Remains 18553033669039720369701110

The last Root 9920700000 × 3 = 29762100000 the new Divisor.

Then 29762100000) 18553033669039720369701110 (*

* 623377841921091 The Quotient or New Resolvend.

Last Root 99207 being more than just, therefore the new Quotients must be subtracted, as in the last Operation.

Thus

Thus	99207)	623377841 921091	(62836,45
	— 6	5952384	
Divisor	992064	28139441	
	— 2	19841276	
Divisor	9920638	8298165	92
	— 8	7936509	76
(*)	99206372	361656	1610
	— 3	297619	1151
	992063717	64037	045991
	— 6	59523	822984
	9920637164	4513	22300700
	— ,4	3968	25486544
	99206371636	544	9681415600
	— ,05	496	0318581775
	9920637163,55		&c.

Last Root 9920700000
— 62836,45 &c.
9920637163,55 the Root required.

Note, *In this Operation all the Figures on this side the Line, and all the new Divisors after the (*) are useless, and might have been omitted.*

Thus I have obtained the Cube Root to twelve Places of Figures, *viz.* 9920637163,55 at two Operations; being but an Unit too much in the last Place of it, as may be tried by involving it to a Cube, and comparing that Cube with the given Resolvend.

In the same manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being first changed into Decimals, may be extracted.

Sect. 4. *To* Extract *the* Biquadrat Root.

In extracting the Biquadrat Root, or that of the fourth Power; (and indeed the Roots of all even Powers) there are some small Difficulties, not so easily express'd and explain'd in a few Words, as they are by an *Algebraic Theorem* (such as shall be shewed further on) I have therefore in this Place, made choice of extracting such Roots by two several Extractions; and the rather, because I presume the Reader by this Time thoroughly acquainted with the Business of extracting the Square Root, by which this may easily be performed. Thus;

First, *Extract the Square Root of the proposed Resolvend, then the Square Root of that first Root will be the Biquadrat Root required.*

Example,

Example 1. What's the Biquadrat Root of 4857532416?
First extract its Square Root,

Thus 4857532416
− 36 = the greatest Square, whose Root is 6.
1257532416 Remainder to be divided by 2.

First Root 6) 628766208 (69696
+ 9 5805
69 4826
+ 6 4158
696 668620
+ 9 626805
6969 418158
418158
(0)

Then 69696 being the first Root, whose Square Root must now be extracted.
− 4
29696 Remainder to be divided by 2.

First Root 2) 14848 (264 the Biquadrat Root as was required.
+ 6 138
26) 1048
+ 4 1048
264 (0)

This is so easy I need not insert any more Examples.

Sect. 5. *To* Extract *the* Sursolid Root.

Having pointed the given Resolvend according as its *Index* denotes; *viz.* into Periods of five Figures; seeking such a Sursolid Number in the *Table of Powers*, or otherwise, as comes the nearest to the first Period of the Resolvend, whether greater or less; and call its respective Root accordingly; *viz. More* than *Just*; or *Less* than *Just*; annexing so many Cyphers to it as there are remaining Periods of whole Numbers in the Resolvend. As before in extracting the Cube Root.

Then find the Difference between the Resolvend, and the Sursolid Number so taken, by subtracting the lesser from the greater (as before in the Cube). Next find the Cube of the aforesaid Sursolid

Surfolid Root with its annexed Cyphers, (which you may alfo do by the *Table of Powers*) and multiply that Cube with 5 the Index of the Surfolid, the Product muft be a Divifor, by which the Difference between the Refolvend and the Surfolid Number muft be divided; that fo it may be depreffed to a Square (as before in the Cube (which muft be pointed into Periods of two Figures each, calling it the new Refolvend, (as before). Then make the firft Root, without its Cyphers, a Divifor enquiring, how oft it may be found in the firft Period of the new Refolvend, with this Confideration, if the Root (now a Divifor) be lefs than juft, you muft annex twice the Quotient Figure to it; but if it be more than juft, you muft fubtract twice the Quotient Figure ftom a Cypher either annexed, or fuppos'd to be annex'd to that Divifor or Root, multiplying it fo encreafed, or diminifhed, with the faid Quotient Figure, fetting down their Product, &c. as before. An *Example* in each Cafe will render it plain and eafy.

Example 1. Suppofe it be required to extract the Surfolid Root out of this Number 12309502009375.

12309502009375 The Refolvend pointed.

The neareft Surfolid Number to 1230, the firft Period of the Refolvend, is 1024, whofe Root is 4, being *lefs* than *juft*.

Therefore 12309502009375
— 1024

2069502009375 their Difference.

Next the Cube of 400 is 64000000 *per Table*, &c.
And 64000000 × 5 = 320000000 the Divifor
Then 320000000) 2069502009375 (6467 &c.

First Root 4) 6467 (15
+ 1 × = 2   42

Divifor 42   2267
+ 5 × 2 = 10   2150

Divifor 430 (117) the Remainder to be rejected:

Firft Root = 400
+ 15

the true Root 415 as required.

That is, 415 is the Surfolid Root of the given Refolvend. As may be eafily tried by involving it to the fifth Power.

Viz. 415 × 415 × 415 × 415 × 415 = 12309502009375 the given Refolvend.

Example.

Example 2. What's the Surſolid Root of 2327834559873

The neareſt Surſolid Number to 232 is 243 whoſe Root is 3 being more than juſt.

Therefore 243000000000
— 232783455987
Remains 10216544012 For a Dividend.

The Cube of 300 is 27000000 And 27000000 × 5 = 135000000

Then 135000000) 102165440127 (756,7810 New Reſolvend.

Firſt Root	300)	756,7810	(2,558
— 02 × 2 =	4	592	
Diviſor	296)	164,78	
— ,5 × 2 =	1.0	147,50	
2. Diviſor.	295,0)	17,2810	
—,05 × 2 =	,10	14,7450	
	294,90	2,53600 &c.	

The firſt Root was 300, being more than juſt.
Therefore it is — 02.558

The n[illegible] Root, 297,442 And is very near the true Root which is 297,436 *&c.* Now the Reaſon why this Root comes out to ſo many Places of Figures at the firſt Operation; is becauſe the firſt Surſolid Number was ſo near the Reſolvend, *&c.* As before.

Sect. 6. *To* Extract *the* Root *of the* Square Cubed.

This may be eaſily performed by two Extractions; according as its Name denotes.

Thus, *Firſt extract the Square Root of the given Reſolvend; then extract the Cube Root of that Square Root: And it will be the Root required. That is, it will be the Root of the ſixth Power.*

Or thus, *Firſt extract the Cube Root of the Reſolvend, then extract the Square Root of that Cube Root: And it will be the Root required.*

Example 1. Let it be required to extract the Square Cubed Root out of this Number 1452205373535515625 the Reſolvend.

Firſt I extract the Square Root of this Reſolvend, which I take to be the beſt and eaſieſt Way.

Thus 145220537353515625
− 9

Remains 55220537353515625 To be halved,

Then 3)	27610268676757812,5 (381078125
+ 8	272
38)	4102
+ 10	3805
3810)	2976867
+ 7	2667245
38107)	3096226
+ 8	3048592
381078)	47634757
+ 1	38107805
3810781)	95269528
+ 2	76215622
38107812)	1905390612,5
+ 5	1905390612,5
381078125	(0)

Having found the Square Root of the given Refolvend, I proceed to extract the Cube Root of that Square Root.

That is, of 381078125
− 343 = the neareft Cube, its Root is 700

Then 700 × 3 = 2100) 38078125 (18161

Firft Root 7)	18161 (25			
+ 2	144			
1. Divifor 72)	3761		Firft Root	700
+ 5	3625		+	25
2. Divifor 725	(136)			725

Hence I find 725 to be the Square Cube Root required; as may eafily be tried by involving it to the fixth Power.

That is, 725 × 725 × 725 × 725 × 725 × 725 will be found = 145220537353515625 the given Refolvend.

Sect.

Sect. 7. *To* Extract *the* Root *of the* Seventh Power.

Having pointed the given Resolvend, as its Index denotes, *viz.* into Periods of 7 Figures, seek out such a Number of the seventh Power, by the *Table of Powers*, as comes nearest to the first Period of the Resolvend; whether it be greater or lesser, calling its respective Root *more* than *just*, or less than *just*, annexing its proper Number of Cyphers, *&c.* as in the Cube and Sursolid.

Then find the Difference between the given Resolvend, and that Number of the seventh Power (found by the *Table of Powers*) by subtracting the lesser from the greater.

Next find the Sursolid or fifth Power of that Root with its annexed Cyphers (which you may also do by the *Table of Powers*) and multiply that Sursolid Number with 7, the Index of the given Resolvend, that Product must be a Divisor, by which the foresaid Difference must be divided; that so it may be depressed to a Square, to be pointed, *&c.* as before in the Cube, *&c.* then make the first Root, without its Cyphers, a Divisor; working with it and the new Resolvend, as before, only here you must encrease, or diminish the Divisor with thrice the Quotient Figure.

Example.

What's the second Sursolid Root, or that of the seventh Power?

of 3732365539550788125 the Resolvend pointed.
— 2178 the nearest Number of the seventh Power.

155436553955078125 their Difference.

The first Root is 300 being less than just; and the fifth Power of 300 is 2430000000000 which being multiplied with 7 is 17010000000000 for a Divisor, by which the aforesaid Difference must be divided; which contracted may stand thus,

1701) 15543655 (9137,95 *&c.*

First Root 3)	9137 (25	First Root	= 300
+ 2 × 3 = 6	72		+ 25
1. Divisor 36	1937		
+ 5 × 3 = 15	1875	True Root	325
375	(62) the Remainder to be rejected as before.		

Hence I have found 325 to be the true Root required, that is, the true Root of the seventh Power.

I think it needless to proceed farther; *viz.* to insert Examples of higher Powers. For if what is already done be well understood, it will be easy to conceive how to proceed in extracting the Root of any single Power how high soever it be (for the Method is general and alike in all Powers) due Regard being had to their Indices; and to the first single Side or Root. That is, whether it be *more*, or *less*, than *just*, &c.

Yet methinks I hear the young Learner say, 'tis possible to follow the Directions and Examples, as they are here laid down; but still here is not the Reason why they are so, and so, perform'd; and why there should be a Remainder left after the true Root is found; *viz.* when the given Resolvend hath a true Root of its kind.

'Tis true, the Reasons of these are not here laid down; neither indeed can they be rendred so plain and intelligible by Words, as by an *Algebraic Process*, from whence the *Theorems* or *Rules* here given, had their first Invention; as shall be shewed in the next Part, when I come to treat of resolving *Compounded* or *Adfected Æquations*; however, take this short and general Account of this Method.

This, and all other of the new Methods of *converging Series* (as they are called) are very different from the former (and still common) Methods of extracting Roots, which requires the first single Side or Root of the first Period (in any Resolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Divisor; which helps to grope out by Trials a second Figure in the Root. And so proceeds on from Point to Point; still repeating the whole Work for every single Figure that comes into the Root. And if by chance there be a Mistake or Error committed in any one Figure (as 'tis possible there may) it spoils the whole Process, which must then be wholly begun anew, or at least from that Part of it where the Error first entred.

But the Nature and Design of the Method which I have here laid down is quite otherwise; it being so contrived, as to gradually lessen the Difference betwixt any proposed Power, and the like Power of another Number assumed; *viz.* it lessens that Difference until it either quite vanishes, or becomes so infinitely small as to be insignificant.

Therefore when any Number is proposed to have its Root extracted; it is here required to take the next nearest Root of the first Period in the Resolvend; that so the Difference betwixt the given Resolvend, and the Homogeneal Power (*viz.* the like

Power)

Power) of the Root thus taken, may be less either in Excess, or Defect. Which Difference being reduced, or depressed lower, becomes so prepared, that by plain Division, comparatively, there will arise such Quotient Figures as will both correct and encrease the first Root to three Places of Figures at least, sometimes to four, or five Places of Figures; according as the said first Difference happens to be more or less; (of which you may have observed Instances): But yet there will be a Remainder left, and perhaps an Excess or Defect in the Root so encreased; *viz.* in the last Figure of it.

Now to rectify the said Excess or Defect in the Root, and to discover whether the given Resolvend be a true figurate Number, or not: That is, whether, it have a true Root of its kind. It will be necessary to make a second Operation; by taking the Root so encreased, and proceeding with it and the given Resolvend, in all respects as in the first Work, like to the third Example of extracting the Cube Root, I say, if the given Resolvend have a true Root, it will appear at this second Operation, and all the aforesaid Differences, *&c.* will vanish; provided the Root required is not to have more than three, or four, Places of Figures in it.

But if the Root be to have more than three Figures in it; or, that the given Resolvend prove to be a Surd Number. Then there will be a Difference as before; which will afford Quotient Figures to rectify and encrease the Root last taken to three times as many Places of Figures, as it had at the Beginning of that second Operation. As you may see in the aforesaid *Example* 3. of the Cube Root; wherein that Root is encreased to twelve Places of Figures at two Operations: which if it were to be extracted the old, and still common, Way, it would require at least forty times the Number of Figures I have here used.

Again, if there chance to be a Mistake committed in any Operation perform'd by the Method here laid down, that Mistake will not destroy the preceding Work, but will be rectified in the next Operation, although it were not discovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root, *&c.* with very little Trouble, if compared with former Methods.

This brief Account, which I have here given, by way of explaining the Nature of this Method of extracting Roots, being well considered and compared with the several Operations of the foregoing Examples, must needs help the Learner to form such an Idea of it, that he cannot, I presume, but understand how to proceed in extracting the Root out of any single Power, how high soever it be; without the Help of an *Algebraic Theorem.*

Not.

Not, but when that comes to be once underſtood; the Work will be much readier and eaſier perform'd: As will appear in the next Part.

I did intend to have here inſerted the whole Buſineſs of Intereſt and Annuities; but finding that it would require too large a Diſcourſe, to ſhew the Grounds and Reaſons of the ſeveral *Theorems* uſeful therein, I have therefore reſerved that Work for the Cloſe of the next Part. Neither indeed can the raiſing of thoſe *Theorems* be ſo well delivered in Words, as by an *Algebraic* Way of *Arguing*; which renders them not only much ſhorter, but alſo plainer and eaſier to be underſtood.

I have alſo omitted that Rule in Arithmetick, uſually called the *Rule* of *Poſition*, or *Rule* of *Falſe*: Becauſe all ſuch Queſtions as can be anſwered by that *Gueſſing Rule*, are much better done by any one, who hath but a very ſmall Smattering of *Algebra*. I ſhall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraic Arithmetick*, wherein I would adviſe the young Learner not to be too haſty in paſſing from one *Rule* to another, and then he will find it very eaſy to be attained.

AN INTRODUCTION TO THE Mathematicks.

PART II.

PROEM.

HAving formerly wrote a ſmall *Tract* of Algebra, perhaps it may ſeem (*to ſome*) very improper to write again upon the ſame Subject; but only (*as the uſual Cuſtom is*) have referr'd my Reader to that Tract. However, becauſe the following Parts of this Treatiſe are managed by an Algebraic Method of Arguing; which may fall into the Hands of thoſe who have not ſeen that Tract, or any other of that kind; I thought it convenient to accommodate the young Geometer with the firſt Elements, or principal Rules, by which all Operations in this Art are performed: That ſo he may not be at a Loſs as he proceeds farther on: Beſides, what I formerly wrote was only a Compendium of that which is here fully handled at large.*

The principal Rules are Addition, Subtraction, Multiplication, Diviſion, Involution, and Evolution; as in common Arithmetick (but differently perform'd): And therefore ſome call it Algebraic Arithmetick. Others call it Arithmetick in Species, becauſe all the *Quantities* concerned in any *Queſtion*, *remain* in their *Subſtituted Letters* (howſoever manag'd by *Addition*, *Subtraction*, or *Multiplication*, &c.) without being *deſtroy'd* or *changed* into others, as *Figures* in common *Arithmetick* are.

Mr. *Harriot* call'd it Logiſtica Specioſa, or *Specious Computation.*

CHAP.

CHAP. I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps.

Sect. 1. *Of* Notation.

THE Method of noting down Letters for Quantities, is various, according to every one's Fancy; but I ſhall here follow the ſame as in my former Tract: And repreſent the Quantity ſought, be it Line, or Number, *&c.* by the ſmall (a), and if more Quantities than one are ſought, repreſent them by the other ſmall Vowels; *e. u.* or *y.*

The given Quantities are repreſented by the ſmall Conſonants, *b. c. d. f. g.* &c.

And for Diſtinction's ſake, mark the Points or Ends of Lines in all Schemes, with the Capital, or great Letters, *viz. A. B. C. D.* &c.

When any Quantity, either given or ſought, is taken more than once, you muſt prefix its Number to it; as $3a$ ſtands for a taken three times, or three times a, and $7b$ ſtands for ſeven times b, *&c.*

All Numbers thus prefixt to any Quantity are called Co-efficients or Fellow-Factors; becauſe they multiply the Quantity; and if any Quantity be without a Co efficient, it is always ſuppos'd, or underſtood to have an Unit prefix'd to it; As a is $1a$, or b is $1b$, &c.

The Signs by which Quantities are chiefly managed are the ſame; and have the ſame Signification, with thoſe in the *Firſt Part, Page* 5. which I here preſume the Reader to be very well acquainted with. To them muſt be here added theſe three more:

Viz. $\left\{\begin{array}{l} \text{@} \\ \omega \\ \surd \end{array}\right\}$ the Sign of $\left\{\begin{array}{l}\text{Involution.} \\ \text{Evolution, or extracting Roots.} \\ \text{Irrationality, or Sign of a Surd Root.}\end{array}\right.$

All Quantities that are expreſs'd by Numbers only, as in Vulgar Arithmetick, are called *Abſolute Numbers.*

Thoſe Quantities that are repreſented by ſingle Letters, as, *a. b. c. d.* &c. or by ſeveral Letters that are immediately joined together; as *ab. cd.* or *7bd.* &c. are called ſimple or ſingle whole Quantities.

But when different Quantities repreſented by different or unlike Letters, are connected together by the Signs (+ or —); As $a+b$. $a-b$. or $ab+dc$. &c. they are called *Compound whole Quantities.*

And

And when Quantities are expreſs'd or ſet down like Vulgar Fractions, Thus $\frac{a}{b}$. Or $\frac{a+b}{d}$. Or $\frac{ab+dc}{b-c}$. &c they are called Fractional or broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concern'd in any Queſtion, may ſtand in any Order at Pleaſure, *viz.* the moſt convenient for the next Operation. As $a+b-d$ may ſtand thus $b-d+a$. Or thus $a-d+b$. Or $-d+a+b$ &c. theſe being ſtill the ſame, tho' differently placed.

That Quantity which hath no Sign before it, as generally the leading Quantity hath not, is always underſtood to have the Sign $+$ before it. As a is $+a$. Or $b-d$ is $+b-d$ &c. for the Sign $+$ is the affirmative Sign, and therefore all leading or poſitive Quantities are underſtood to have it, as well as thoſe that are to be added.

But the Sign — being the negative Sign, or Sign of Defect, there is a Neceſſity of prefixing it before that Quantity to which it belongs, where-ever the Quantity ſtands.

Sect. 2. *Of* Tracing *the* Steps *uſed in bringing the the* Quantities *to an* Equation.

The Method of Tracing the Steps, uſed in bringing the Quantities concern'd in any Queſtion to an Æquation, is beſt performed by regiſtring the ſeveral Operations, with Figures and Signs placed in the Margin of the Work, according as the ſeveral Operations require; being very uſeful in long and tedious Operations.

For Inſtance: If it be required to ſet down, and regiſter the Sum of the two Quantities a, and b, the Work will ſtand,

Thus		
	1	a
	2	b
$1+2$	3	$a+b$

Firſt ſet down the propoſed Quantities, a and b over againſt the Figures 1. 2. in the ſmall Column, which are here called Steps, and againſt 3, the third Step, ſet down their Sum, *viz.* $a+b$. Then againſt that third Step, ſet down $1+2$ in the Margin; which denotes that the Quantities againſt the firſt and ſecond Steps are added together, and that thoſe in the third Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe $a=9$ and $b=6$. Then it will be,

Thus		
	1	$a=9$
	2	$b=6$
$1+2$	3	$a+b=9+6=15$

being the Sum of 9 and 6.

Again, If it were required to ſet down the Difference of the ſame two Quantities, Then it will be,

Thus	1	$a = 9$
	2	$b = 6$
$1 - 2$	3	$a - b = 9 - 6 = 3$ the Difference between 9 and 6.

Or if it were required to ſet down their Product.
Then it will be,

Thus	1	$a = 9$
	2	$b = 6$
1×2	3	$a \times b$ or $ab = 9 \times 6 = 54$ the Product of 9 into 6.

&c.

Note, Letters ſet or joyn'd immediately together, like a Word, ſignifie the Rectangle or Product of thoſe Quantities they repreſent. As in the laſt *Example*, wherein $ab = 54$ is the Product of $a = 9$ and $b = 6$. &c.

Axioms.

1. If *Equal Quantities* be **Added** to *Equal Quantities*, the **Sum** of thoſe *Quantities* will be equal.
2. If *Equal Quantities* be **Taken** from *Equal Quantities*, the *Quantities* **Remaining** will be *equal*.
3. If *Equal Quantities* be **Multiplied** with *Equal Quantities*, their **Products** will be *equal*.
4. If *Equal Quantities* be **Divided** by *Equal Quantities*, their **Quotients** will be *equal*.
5. Thoſe *Quantities*, that are *Equal* to one and the ſame *Thing*, are *Equal* to one *another*.

Note, *I adviſe the* Learner *to get theſe five* **Axioms** *perfectly by Heart.*

Theſe things being premiſed, and a perfect Knowledge of the Signs, and their Significations being gained, the young *Algebraiſt* may proceed to the following Rules. But firſt I muſt make bold to adviſe him here, as I have formerly done, that he be very ready in one Rule before he undertakes the next.

That is, he ſhould be expert in Addition, before he meddles with Subtraction; and in Subtraction, before he undertakes Multiplication, &c. becauſe they have a Dependency one upon another.

CHAP.

CHAP. II.

Concerning the Six *Principal* Rules of Algebraic Arithmetick, of whole Quantities.

Sect. 1. Addition of whole Quantities.

Addition of whole Quantities admits of *Three Cases*.

Case 1. If the *Quantities* are *like*, and have *like Signs*; *Add* the *Co-efficients or prefixt Numbers* together; and to their *Sum* adjoyn the *Quantities* with the same *Sign*.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	a	$-a$	$5b$	$-7bc$
	2	a	$-a$	$3b$	$-8bc$
1 + 2	3	$2a$	$-2a$	$8b$	$-15bc$

Thus		Exam. 5.	Exam. 6.	Exam. 7.
	1	$3a+5b$	$3a-5b$	$6ab+12$
	2	$2a+7b$	$2a-7b$	$3ab+24$
1 + 2	3	$5a+12b$	$5a-12b$	$9ab+36$

The Reason of these Additions is evident from the Work of common Arithmetick. For suppose a. to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or $2a$. As in *Example* 1.

Or if we suppose $-a$. to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns, or $-2a$; as in *Example* 2. And so for all the rest.

Case 2. If the *Quantities* are *alike*, and have *unlike* Signs, subtract the *Co-efficients* from each other, and to their Difference joyn the *Quantities* with the Sign of the *Greater*.

		Exam. 8.	Exam. 9.	Exam. 10.	Exam 11.
	1	$+5a$	$-5a$	$7bc$	$-9abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
1 + 2	3	$+2a$	$-2a$	bc	$-2abd$

		Exam. 12.	Exam. 13.
	1	$7a-5b$	$-8ab-7bc+15$
	2	$-5a+7b$	$+12ab+7bc-24$
1 + 2	3	$2a+2b$	$4ab-9$

The Reaſon of the Operations in this Caſe may be eaſily underſtood by any one that duly conſiders the comparing of Stock and Debts together; or the Ballancing of Accompts betwixt Debtor and Creditor.

That is, the Affirmative Quantities repreſent the Stock or Creditor: The Negative Quantities repreſent the Debts; and their Sum repreſents the Ballance, *&c.*

Caſe 3. When the Quantities are unlike, ſet them all down without altering their Signs; and thence will ariſe compound Quantities, which can be no otherwiſe added but by their Signs.

Thus	1	a	a	$5b+7dc$
	2	b	$-b$	$4a-20$
$1+2$	3	$a+b$	$a-b$	$5b+7dc+4a-20$

Here follow ſome few Examples wherein all the 3 Caſes are promiſcuouſly concerned.

	1	$aa+2ab+bb$	$8ab+bc-37$
	2	$-4ab$	$-7ab-bc+42-6d$
$1+2$	3	$aa-2ab+bb$	$ab+5-6d$

	1	$aa-2ab+bb$	$9bc+7ab-45$
	2	$+4ab+bb$	$4d-6bc-7ab+da$
$1+2$	3	$aa+2ab+bb$	$3bc+4d-45+da$

	1	$5a$	$a+b-ab$
	2	$-7a$	$7c-d$
	3	$+3a$	$4e+f$
$1+2+3$	4	a	$a+b-ab+7c-d+4e+f$

	1	$3aa+4abc-bb+30$
	2	$2bb-3aa+2abc-25$
	3	$dd+2aa-5abc-3$
$1+2+3$	4	$bb+dd+2aa+abc+2$

Sect. 2. Subtraction of whole Quantities.

Subtraction of whole Quantities is perform'd by one general *Rule.*

Rule.

Change all the Signs of the Subtrahend; viz. *of thoſe Quantities which are to be ſubtracted, or ſuppoſe them in your Mind to be changed. Then add all the Quantities together, as before in Addition, and their* Sum *will be the true Remainder or Difference required.*

This

This *General Rule* is deduced from these evident Truths.

To subtract an Affirmative Quantity, from an Affirmative; is the same as to add a Negative Quantity to an Affirmative.

That is, $+2a$ taken from $+3a$, is the same with $-2a$ added to $+3a$.

Consequently, To subtract a Negative Quantity from an Affirmative; will be the same as to add an Affirmative Quantity to an Affirmative.

That is $-2a$ taken from $+3a$ will be the same with $+2a$ added to $+3a$.

		Exam. 1.	*Exam* 2.	*Exam.* 3.	*Exam.* 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$3b$	$-8bc$
1 — 2	3	a	$-a$	$5b$	$-7bc$

		Exam. 5.	*Exam.* 6.	*Exam.* 7.
	1	$5a+12b$	$5a-12b$	$9ab+36$
	2	$2a+7b$	$2a-7b$	$3ab+24$
1 — 2	3	$3a+5b$	$3a-5b$	$6ab+12$

		Exam. 8.	*Exam.* 9.	*Exam.* 10.	*Exam.* 11.
	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
1 — 2	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	*Exam.* 13.
	1	$2a+2b$	$4ab-9$
	2	$-5a+7b$	$-8ab-7bc+15$
1 — 2	3	$7a-5b$	$12ab+7bc-24$

If these 13 Examples be compared with those in Addition; the Work will appear very evident, these being only the Converse or Proof of those; according to the Nature of Addition and Subtraction, in common Arithmetick.

More *Examples* in *Subtraction*.

	1	$a+b$	$5bc+3da$	$8a+5bd+25$
	2	$a-b$	$5bc-4da$	$7a-3bd-12$
1 — 2	3	$+2b$	$+7da$	$a+8bd+37$

	1	$c+13$	a	0
	2	$3a-b-2c$	b	$2a-4b$
$1-2$	3	$3c+13-3a+b$	$a-b$	$-2a+4b$

	1	$a+b-54$	76
	2	$d-3b-bc-75$	$a-b-5d+7c$
$1-2$	3	$a+4b+bc+21-d$	$76-a+b+5d-7c$

That, $a-b$ taken from $a+b$ leaves $+2b$ for the Remainder; as in the firſt of theſe Examples, may be thus proved.

Let	1	$a+b=z$
And	2	$a-b=x$
$2+b$	3	$a=x+b.$ *per* Axiom 1.
$1-3$	4	$b=z-x-b.$ *per* Axiom 2.
$4+b$	5	$2b=z-x.$ which was to be proved.

The Truth of all Operations in Subtraction, where any Doubt ariſes, may be proved, by adding the Subtrahend to the Remainder; as in common Arithmetick.

Examples.

From	1	$+5a$	0	$-9bc$	
Take	2	$-2a$	$+3b$	$-6da$	*Subtrahend.*
$1-2$	3	$+7a$	$-3b$	$+6da-9bc$	*Remainder.*
$2+3$	4	$+5a$	0	$-9bc$	*Proof.*

Sect. 3. **Multiplication** of whole **Quantities.**

Multiplication of whole Quantities admits of *Three Caſes.*

Caſe 1. When the Quantities have like Signs, and no Co-efficients, ſet or joyn them together; and prefix the Sign $+$ before them, and that will be their Product.

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Thus	1	a	$-a$	$a+b$	$-a-b$
	2	b	$-b$	d	$-d$
$1+2$	3	ab	$+ab$	$ad+bd$	$+ad+bd$

Caſe. 2. If there be Co-efficients; multiply them, and to their Product adjoyn the Quantities ſet together as before.

Thus

Thus		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
	1	$5a$	$-6d$	$3a+2b$	$a+b$
	2	$3b$	$-7b$	6	$5b$
1 × 2	3	$15ab$	$+42db$	$18a+12b$	$5ab+5bb$

Case 3. When the Quantities have unlike Signs; joyn them and the Product of their Co-efficients together (as before) but prefix the Sign $-$ before them;

Thus		Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
	1	$+a$	$-6d$	$4a-7b$	$4a-7b$
	2	$-b$	$+7b$	$3f$	$-3f$
1 × 2	3	$-ab$	$-42db$	$12af-21bf$	$-12af+21bf$

That is, $+$ into $+$, or $-$ into $-$, gives $\left.\begin{matrix}+\\-\end{matrix}\right\}$ in the Product.
But $+$ into $-$, or $-$ into $+$, gives

That $+$ into $+$, will produce $+$ in the Product, is evident from Multiplication in common Arithmetick-
viz. $+5$ into $+7$ will give $+35$, *&c*.
But that $+$ into $-$, Or $-$ into $+$ should produce the Sign $-$, As in the four last Examples.
And that $-$ into $-$ should produce the Sign $+$ as in the second, fourth, and sixth Examples, may perhaps seem somewhat hard to be conceived; and requires a Demonstration.

First to prove that $-7b$ into $+3f = -21bf$. As in *Ex.* 11.

Suppose	1	$4a-7b=0$	
Then will	2	$4a=7b$.	*per* Axiom 1.
But	3	$+3f=+3f$	
2 × 3	4	$12af=21bf$.	*per* Axiom. 3.
4 − 21bf	5	$12af-21bf=0$	*per* Axiom 2.

Consequently $+$ into $-$, Or $-$ into $+$ produces $-$, which was the thing to be proved.

Secondly, to prove $-7b$ into $-3f$ gives $+21bf$ as in *Exam.* 12.

Let	1	$4a-7b=0$	as before.
Then	2	$4a=7b$	
But	3	$-3f=-3f$	
the 2 × 3 is	4	$-12af=-21bf$	by what is proved above.
4 + 21bf	5	$-12af+21bf=0$.	*per* Axiom 1.

Consequently $-$ into $-$ gives $+$ which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, Suppose $\left\{\begin{matrix} a = 20 \\ b = 14 \end{matrix}\right\}$ and $\left\{\begin{matrix} c = 12 \\ d = 8 \end{matrix}\right\}$ $\left\{\begin{matrix}\text{or any} \\ \text{other Numbers.}\end{matrix}\right.$

Then $a - b = 6$ $c - d = 4$ *per Axiom* 2.

Consequently, $a - b \times c - d = 6 \times 4 = 24$ *per Axiom* 3. but $a - b \times c - d$ according to the Preceding Rules, will be, $ac - cb + bd - da$, which if true must be equal to 24.

Proof $\left\{\begin{matrix} ac = 20 \times 12 = 240 \\ bd = 14 \times 8 = 112 \end{matrix}\right.$ $\begin{matrix} cb = 12 \times 14 = 168 \\ da = 8 \times 20 = 160 \end{matrix}$

Hence, $ac + bd = 352$ *per Axiom* 1.
And $cb + da = 328$ which being subtracted,
Leaves $ac + bd - cb - da = 352 - 328 = 24$ which plainly shews,

That $+$ into $-$ Produces $-$ } in the Product.
And $-$ into $-$ Produces $+$ }

Q. E. D.

Note. *If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand: And the Sum of those particular Products, will be the Product requir'd. As in common Arithmetick.*

Examples.

	1	$a + b - d$	$7b + 5d$
	2	$a - b$	$3a - 5f$
$1 \times a$	3	$aa + ba - da$	$21ba + 15da$
$1 \times b$	4	$- ba - bb + db$	$- 35bf - 25df$
$3 + 4$	5	$aa - da - bb + db$	$21ba + 15da - 35bf - 25df$

	1	$aa - ba$	$2c - 3d$
	2	$a + b$	$3a - 4b$
1×2	3	$aaa - abb$	$6ca - 9da - 8bc + 12db$

	1	$aa + 2a + 4$	$aa - ba + bb$
	2	$a - 2$	$a + b$
		$aaa + 2aa + 4a$	$aaa - baa + bba$
		$- 2aa - 4a - 8$	$+ baa - bba + bbb$
1×2	3	$aaa - 8$	$aaa + bbb$

Sect. 4. Diviſion of whole Quantities.

Diviſion of Species, is the converſe or direct contrary to that of Multiplication, and conſequently perform'd by converſe Operations (As in common Arithmetick) and admits of four *Caſes*.

Caſe 1. When the Quantities in the Dividend, have like Signs to thoſe in the Diviſor, and no Co-efficients in either; caſt off or expunge all the Quantities in the Dividend, that are like thoſe in the Diviſor; and ſet down the other Quantities with the Sign + for the Quotient required.

Thus					
	1	ab	$-ab$	$ad+bd$	$-ad-bd$
	2	b	$-b$	d	$-d$
1 ÷ 2	3	a	$+a$	$a+b$	$a+b$

Caſe 2. When the Quantities in the Dividend have unlike Signs to thoſe in the Diviſor; then ſet down the Quotient Quantities found as before, with the Sign − before them.

Thus				
	1	$+ab$	$-ab-bd$	$abc+bcd+bcf$
	2	$-b$	$+b$	$-bc$
1 ÷ 2	3	$-a$	$-a-d$	$-a-d-f$

Caſe 3. If the Quantities in the Dividend and Diviſor, have Co-efficients; Divide the Numbers (as in common Arithmetick) and to their Quotients adjoin the Quotient Quantities.

Thus				
	1	$15ab$	$42db$	$12af-21bf$
	2	$3b$	$-7b$	$3f$
1 ÷ 2	3	$5a$	$-6d$	$4a-7b$

Note. When the Quantities and Co-efficients in the Diviſor and Dividend are all the ſame, the Quotient will be an Unit or 1.

Thus					
	1	ab	$9bc$	$7ab+5bc$	$8ab+4d$
	2	ab	$-9bc$	$7ab+5bc$	$-8ab-4d$
1 ÷ 2	3	1	-1	1	-1

Caſe 4. When the Quantities in the Diviſor cannot be exactly found in the Dividend; then ſet them both down like a Vulgar Fraction. As in common Arithmetick.

Thus	1	a	$6bc$	$5b+aa$	$8adc$
	2	b	$3d$	$5d+7b$	$4abc$
$1 \div 2$	3	$\frac{a}{b}$	$\frac{2bc}{d}$	$\frac{5b+aa}{5d+7b}$	$\frac{2d}{b}$

N. B. *In Division one thing must be very carefully observed*; viz. *that like Signs give* + *and unlike Signs give* — *in the Quotient*; *which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning Multiplication and Division, in Vulgar Arithmetick.*

Examples of Division at large.

	1	$21ba + 15da - 35bf - 25df \quad (+\, 3a$
	2	$7b + 5d$
$2 \times 3a$	3	$21ba + 15da$
$1 - 3$	4	$0 \qquad 0 - 35bf - 25df \quad (-\, 5f$
$2 \times -5f$	5	$-35bf - 25df$
$4 - 5$	6	$0 \qquad 0$
$1 \div 2$	7	$3a - 5f$ the Quotient collected from the 3. and 5. Steps

Or Division of Quantities may stand as Numbers in common Arithmetick do; Thus

$$
\begin{array}{l}
3a - 6\,)\ 6aaaa - 96 \quad (\,2aaa + 4aa + 8a + 16 \\
\qquad\quad 6aaaa - 12aaa \\
\qquad\qquad 0 + 12aaa - 96 \\
\qquad\qquad\quad + 12aaa - 24aa \\
\qquad\qquad\qquad 0 + 24aa - 96 \\
\qquad\qquad\qquad\quad + 24aa - 48a \\
\qquad\qquad\qquad\qquad 0 + 48a - 96 \\
\qquad\qquad\qquad\qquad\quad + 48a - 96 \\
\qquad\qquad\qquad\qquad\qquad 0 \quad 0
\end{array}
$$

That is, $6aaaa - 96 \div 3a - 6$ gives $2aaa + 4aa + 8a + 16$ for the Quotient, as may easily be proved by Multiplication, *viz.* $2aaa + 4aa + 8a + 16 \times 3a - 6$ will produce $6a^4 - 96$ and so for the rest.

Sect. 6. Involution of whole Quantities.

Involution is the raising or producing of Powers, from any proposed Root, and is performed in all respects like Multiplication save only in this; Multiplication admits of any different Factors, but Involution still retains the same.

Exam.

Examples.

	1	a	$-a$	the Root, or single Power.
1 ⊕ 2	2	aa	$+aa$	Square, or Second Power.
1 ⊕ 3	3	aaa	$-aaa$	Cube, or third Power.
1 ⊕ 4	4	$aaaa$	$+aaaa$	Biquadrat, or 4th Power.
1 ⊕ 5	5	$aaaaa$	$-aaaaa$	Sursolid, or 5th Power, &c.

Note, *The Figures placed in the Margin, after the Sign* (⊕) *of Involution; shew to what Height the Root is involved; and are called Indices of the Power; and are usually placed over the involved Quantities, in order to contract the Work, especially when the Powers are any thing high.*

Thus $\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases}$ And $\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5\ b^6 = aaaaabbbbbb \\ a^3\ b^2\ d^4 = aaabbdddd \end{cases}$

If the Quantities have Co-efficients, the Co-efficients must be involved along with the Quantities. As in these,

Thus				
	1	$2a$	$-\ 3a$	$5bc$
1 ⊕ 2	2	$4aa$	$+\ 9aa$	$25bbcc$
1 ⊕ 3	3	$8aaa$	$-\ 27aaa$	$125bbbccc$
1 ⊕ 4	4	$16aaaa$	$+\ 81aaaa$	$625b^4c^4$
1 ⊕ 5	5	$32aaaaa$	$-\ 243a^5$	$3125b^5c^5$ &c.

Involution of compound Quantities is performed in the same manner, due Regard being had to their Signs and Co-efficients, if there be any.

As for Instance, Suppose $a+b$ were given to be involved to the 5th Power.

Thus			
	1	$a+b$	called a Binomial Root.
		$a+b$	
1 × a	2	$aa+ab$	
1 × b	3	$\quad +ab+bb$	
1 ⊕ 2	4	$aa+2ab+bb$	the Square of $a+b$
		$a+b$	
4 × a	5	$aaa+2aab+abb$	
4 × b	6	$\quad +aab+2abb+bbb$	
1 ⊕ 3	7	$aaa+3aab+3abb+bbb$	the Cube of $a+b$.

	7	$aaa + 3aab + 3abb + bbb$
		$a + b$
$7 \times a$	8	$a^4 + 3a^3b + 3aabb + abbb$
$7 \times b$	9	$\quad + a^3b + 3aabb + 3abbb + b^4$
1 ⊛ 4	10	$a^4 + 4a^3b + 6aabb + 4abbb + b^4$
		$a + b$
$10 \times a$	11	$a^5 + 4a^4b + 6a^3bb + 4aab^3 + ab^4$
$10 \times b$	12	$\quad + a^4b + 4a^3bb + 6aab^3 + 4ab^4 + b^5$
1 ⊛ 5	13	$a^5 + 5a^4b + 10a^3bb + 10aab^3 + 5ab^4 + b^5$
		$a + b$ &c.

Again, Let $a - b$, called a Refidual Root, be given.

Then	1	$a - b$
		$a - b$
$1 \times a$	2	$aa - ab$
$1 \times -b$	3	$\quad - ab + bb$
1 ⊛ 2	4	$aa - 2ab + bb$ the Square of $a - b$
		$a - b$
$4 \times a$	5	$aaa - 2aab + abb$
$4 \times -b$	6	$\quad - aab + 2abb - bbb$
1 ⊛ 3	7	$aaa - 3aab + 3abb - bbb$ the Cube of $a - b$.
		$a - b$
$7 \times a$	8	$aaaa - 3aaab + 3aabb - abbb$
$7 \times -b$	9	$\quad - aaab + 3aabb - 3abbb + bbbb$
1 ⊛ 4	10	$aaaa - 4aaab + 6aabb - 4abbb + bbbb$
		$a - b$
$10 \times a$	11	$a^5 - 4a^4b + 6a^3bb - 4aab^3 + ab^4$
$10 \times -b$	12	$\quad - a^4b + 4a^3bb - 6aab^3 + 4ab^4 - b^5$
1 ⊛ 5	13	$a^5 - 5a^4b + 10a^3bb - 10aab^3 + 5ab^4 - b^5$
		$a - b$ &c.

By comparing thefe two Examples together, you may make the following Obfervations.

1. That the Powers raifed from a Refidual Root (*viz.* the Difference of two Quantities) are the fame with their like Powers raifed from a Binomial Root (or the Sum of two Quantities) fave only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term; but the Refidual Powers have the Signs + and − interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity (a)

ty (a) continually decreaſe in Arithmetical Progreſſion; *viz.* in the Square it is aa, a, In the Cube aaa, aa, a.

In the Biquadrat it is, $aaaa$, aaa, aa, a, &c.

3. The Indices of the other Quantity b do continually encreaſe in Arithmetical Progreſſion; *viz.* In the Square it is b, bb. In the Cube b, bb, bbb. In the Biquadrat it is b, bb, bbb, $bbbb$. &c.

4. The firſt and laſt Terms, are always pure Powers of the ſingle Quantities, and are both of the ſame Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the higheſt Power, *viz.* of the firſt or laſt Term.

These Obſervations being duly conſidered, it will be eaſy to conceive how the Terms of any propoſed Power raiſed from a Binomial or Reſidual Root muſt ſtand, without their *Unciæ* or Numeral Figures.

For Inſtance, ſuppoſe it were required to raiſe the Binomial Root $a+b$ to the ſeventh Power; then the Terms of that Power will ſtand without their *Unciæ* in this Order.

Viz. $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$.

And becauſe the *Uncia* (not only of any ſingle Letter, but alſo) of every ſingle Power, how high ſoever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial, or Reſidual Root are naturally raiſed by multiplying of the preceding Power into its original Root, which is done by only joining each Letter in the Root to the preceding Power, with its *Uncia*, and then removing the ſaid Power, when it is ſo join'd to the ſecond Letter, one Place forwards (either to the left, or right Hand) it muſt needs follow,

That the *Unciæ* of the ſecond Terms (in any ſuch Power) will always be the Sum of ſo many Units added together more one, as there have been Multiplications of the firſt Root; which will always be determined by the Index of the firſt Term in the Power.

And becauſe the *Unciæ* of all the intermediate Terms, are only removed along with their Letters, it alſo follows; that if they are added together, their reſpective Sums will produce the true *Unciæ* of the intermediate Terms in the new raiſed Power. As doth plainly appear from the following Numbers ſo removed without their Letters; which both ſhews and demonſtrates an eaſy Way of producing the *Unciæ* of any ordinary Power (*viz.* of one not very high) raiſed from either a Binomial, or Reſidual Root.

Thus

Thus

Add	1 . 1 . 1 . 1	The two *Unciæ* of the first Root.
Add	1 . 2 . 1 1 . 2 . 1	The *Unciæ* of the Square.
Add	1 . 3 . 3 . 1 1 . 3 . 3 . 1	The *Unciæ* of the Cube.
Add	1 . 4 . 6 . 4 . 1 1 . 4 . 6 . 4 . 1	The *Unciæ* of the 4th Power.
Add	1 . 5 . 10 . 10 . 5 . 1 1 . 5 . 10 . 10 . 5 . 1	*Unciæ* of the 5th Power.
Add	1 . 6 . 15 . 20 . 15 . 6 . 1 1 . 6 . 15 . 20 . 15 . 6 . 1	*Unciæ* of 6th Power.
	1 . 7 . 21 . 35 . 35 . 21 . 7 . 1	*Unciæ* of 7 Power.

And so on in this manner *ad infinitum*.

Now if these Numbers are prefix'd to the aforesaid Letters, all the Terms will be completed with their respective *Unciæ*, and will stand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

But that the Business of finding these *Unciæ*, may be rendred yet more easy for Practice, it will be convenient to consider what Series or Progression, the *Unciæ* of each Term do make, from the aforesaid Additions.

Unciæ of the First Term.	*Unciæ of the Second Term.*	*Unciæ of the Third Term.*	*Unciæ of the Fourth Term.*	*Unciæ of the Fifth Term.*	*Unciæ of the Sixth Term.*	*Unciæ of the Seventh Term.*	*Unciæ of the Eighth Term. &c.*	
1	1							*Unciæ of the single Quantities.*
1	2	1						*Unciæ of the Square.*
1	3	3	1					*Unciæ of the Cube.*
1	4	6	4	1				*Unciæ of the Fourth Power.*
1	5	10	10	5	1			*Unciæ of the Fifth Power.*
1	6	15	20	15	6	1		*Unciæ of the Sixth Power.*
1	7	21	35	35	21	7	1	*Unciæ of the Seventh Power, &c.*

The *Unciæ* of the first Term are only a Series of Units, whose Sum is every where the *Unciæ* of the second Term.

The *Unciæ* of the second Term are a Series of Numbers in Arithmetical Progression; whose Sum is every where the *Unciæ* of the next superior Power in the third Term, and may be found by *Proposition* 1. Chap. 6. Part 1.

That

That is, in the 7th Power it will be the $\frac{6+1\times 6}{2}=21$ *Uncia* of the third Term.

The rest of the *Unciæ* are a compounded Series, whose respective Sums may be obtained from the *Unciæ* of their preceding Terms.

Thus $\frac{21\times 5}{3}=35$. Then $\frac{35\times 4}{4}=35$. Again $\frac{35\times 3}{5}=21$

And $\frac{21\times 2}{6}=7$, &c.

From hence may be deduced this *General Rule.*

Rule.

If the Index of the First Letter of any Term, be multiplied into its own Uncia, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Uncia of the next succeeding Term forward.

That is, by the Help of those Indices that belong to the several Powers of the first or leading Letter only (as a) the true *Uncia* of every Term may be easily found.

Examples.

Let it be required to complete all the Terms of the aforesaid several Powers, viz. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$. with their proper *Uncia.*

1. The Index of a^7 the first Term will be the *Uncia* of the second Term. Thus a^7+7a^6b.

2. Then half the Second Term's Index into its *Uncia*, *Viz.* $\frac{7\times 6}{2}=21$ will be the third Term's *Uncia.*

Thus $a^7+7a^6b+21a^5b^2$ will be the three first Terms.

3. Again $\frac{21\times 5}{3}=35$ is the *Uncia* of the fourth Term.

Then it will be $a^7+7a^6b+21a^5b^2+35a^4b^3$.

3. And $\frac{35\times 4}{4}=35$ will be the *Uncia* of the fifth Term.

Then $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$, &c. until all the Terms are completed with their respective *Unciæ*; and then they will stand

Thus. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

Now

Now here it may be further obſerv'd, that the *Unciæ* do only encreaſe until the Indices of the two Letters become equal, or change Places; and then the reſt of the *Unciæ* will return or decreaſe in the ſame Order. That is, wherever the Indices of the Letters are alike, there the *Unciæ* will be alike.

And therefore one needs to find the *Unciæ* (as before) but to half the Number of Terms in any Power.

If what hath been here ſaid, and the Work of the Example be well underſtood, I preſume it will be found very eaſy to raiſe any Power from a binomial or reſidual Root, to what Height you pleaſe; without the Trouble of a continued Involution; and without the Help of ſuch a Table of Powers as is propoſed by Mr. *Oughtred* in his *Key* to the *Mathematicks*. Page 40. and ſince by others.

Now from theſe Conſiderations it was, that I propoſed this Method of raiſing Powers in my *Compendium* of *Algebra*. Page 57. as wholly new (viz. ſo much of it as was there uſeful) having then, (I profeſs) neither ſeen the Way of doing it, nor ſo much as heard of its being done. But ſince the writing of that *Tract*, I find in Dr. *Wallis*'s *Hiſtory of Algebra*, Page 319 and 331, that the learned Sir *Iſaac Newton* had diſcovered it long before; which the *Doctor* ſets down in this Manner.

Let m be the Exponent of the Power.

$$\text{Then}\left\{\; 1\times\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\;\&c.\right.$$

Will be the Series of the *Unciæ* required; but he doth not tell us how they firſt came to be found out, nor have I ever met with the leaſt Hint of it in any Author.

Sect. 6. Evolution of whole Quantities.

Evolution is the extracting of Roots from any given Power. That is, it is the converſe Work to that of Involution. and in ſingle Quantities 'tis eaſy, if the given Power hath ſuch a Root as is required, which may be thus known.

If the given Power hath no Numbers prefix'd to it, and its Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root ſought.

Thus, if the Cube Root of $aaaaaa$ viz. a^6 were required (the Index of the Cube is 3) then 3) 6 (2. That is, 3) a^6 (a^2 the Root required. And ſuch Operations are uſually ſet down Thus

Thus	1	a^6	a^6b^6	$a^6b^6d^6$
1 ω 2	2	a^3	a^3b^3	$a^3b^3d^3$
1 ω 3	3	aa	$aabb$	$aabbdd$
3 ω 2	4	a	ab	abd

Note, *The Figures placed in the Margin after the Sign* (ω) *of Evolution; denote the Index of the Root to be extracted.*

If the given Powers have Co-efficients: *viz.* Numbers prefix'd to them) then you must extract the respective Roots as in Vulgar Arithmetick.

Thus	1	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
1 ω 2	2	$9aa$	$36a^4b^4$	$144aabbcc$
1 ω 4	3	$3a$	$6aabb$	$12abc$
Or 2 ω 2	4	$3a$	$6aabb$	$12abc$

But if the Root required cannot be truly extracted out of both the Co-efficients and Indices of the given Power; then it is a Surd, and must have the Sign of the Root required prefix'd to it.

Thus	1	a^5	$67aaaa$	$216bbbddd$
1 ω 2	2	$\sqrt{a^5}$	$\sqrt{67aaaa}$	$\sqrt{216bbbddd}$
1 ω 3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67aaaa}$	$6bd$

Evolution of Compound Quantities or Powers, is a little more troublesom than that of single Powers; and would require a great many Words to explain the Manner, and Reason of forming the several Canons, that are commonly used in extracting the Roots of Compound Quantities; especially if the Powers be very high, &c. I shall therefore for Brevity's sake omit them, and instead thereof propose an easy Method of discovering the Roots of all compound Powers in general; and in order to that, it will be necessary to premise; that if either the Sum or Difference of several Quantities be involved to any Power, there will arise so many single Powers of the same Height, as there are different Quantities.

As for Instance, if $a+b+d$ be squared; that is, be involved to the second Power, it will be $aa+2ab+2ad+bb+2bd+dd$, here you have aa, bb, and dd.

Y

Again,

Again, if $a + b + d$ were cubed, *viz.* involved to the third Power, then you will have aaa, bbb, and ddd, in it, *&c.*

Whence it follows, that in extracting the Roots of all compound Quantities, there must be consider'd,

1. How many different Letters, or Quantities, there are in the given Power.

2. Whether the single Powers of each of those Letters be of an equal Height, and have in them such a single Root as is required: Which if they have, extract it as before.

3. Connect those single Roots together with the Sign $+$ and involve them to the same Height with the given Power; that being done, compare the new raised Power with the given Power, and if they are alike in all their respective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be easily corrected with the Sign $-$ as Occasion requires.

Example 1. Let it be required to extract the Square Root of $cc + 2cb - 2cd + bb - 2bd + dd$.

In this compound Square there are three distinct Powers, *viz.* bb, cc, dd, whose single Roots are b, c, d, wherefore I suppose the Root sought to be $b + c + d$, or rather $b + c - d$, because in the given Power there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$, then $b+c-d$ being squared, produces $bb+2bc-2bd+cc-2cd+dd$ which I find to be the same in all its Terms with the given Power, although they stand in a different Position; consequently $b+c-d$ is the true Root required.

Example 2. 'Tis required to extract the Square Root of $a^4-2aabb+b^4$ Here are but two single Powers, *viz.* a^4 and b^4, whose Square Roots are aa, and bb. And because in the given Power there is $-2aabb$, therefore I conclude it must either be $aa-bb$, or $bb-aa$. Both which being involved, will produce $a^4 - 2aabb + b^4$ consequently the Root sought may either be $aa-bb$ or $bb-aa$ according to the Nature or Design of the Question, from whence the given Power was produced.

Example 3. Let it be required to extract the Square Root of $36aaaa + 108aa + 81$. Here the two single Powers are $36aaaa$, and 81, whose Roots are $6aa$ and 9. And because the Signs are all $+$ therefore I suppose the Root to be $6aa + 9$, the which being involved doth produce $36a^4 + 108aa + 81$, consequently $6aa + 9$ is the true Root required.

Example

Example 4. Suppose it were required to extract the Cube Root of $125aaa + 300aae - 450aa + 240aee - 720ae + 64eee + 540a - 288ee + 432e - 216$.

In this Example there are three distinct Powers, *viz.* $125aaa$, $64eee$, and -216.

The Cube Root of $125aaa$ is $5a$, of $64eee$ is $4e$, and the Cube Root of -216 is -6.

Wherefore I suppose the Root sought to be $5a + 4e - 6$, which being involved to the third Power, does produce the same with the given Power, consequently $5a + 4e - 6$ is the Cube Root required.

But if the new Power raised from the supposed Root (being involved to its due Height) should not prove the same with the given Power, *viz.* if it hath either more or fewer Terms in it, *&c.* Then you may conclude the given Power to be a Surd, and must have its proper Sign prefix'd to it, and cannot be otherwise express'd, until it come to be evolved in Numbers.

Example 5. Suppose it were required to extract the Cube Root of $27aaa + 54baa - 8bbb$.

Here are two distinct and perfect Cubes, *viz.* $27aaa$, and $8bbb$, whose Cube Roots are $3a$ and $2b$.

Wherefore one may suppose the Root sought to be $3a + 2b$ which being involved to the third Power, is $27aaa + 54baa + 36bba + 8bbb$. Now this new raised Power hath one Term (*viz.* $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not so, *viz.* it is a Surd, and hath not such a Root as was required, but must be express'd or set down,

$$\text{Thus} \quad \sqrt[3]{27aaa + 54baa + 8bbb}.$$

If these Examples be well understood, the Learner will find it very easy by this Method of proceeding to discover the true Root of any given Power whatsoever.

CHAP. III.

Of Algebraic Fractions, or Broken Quantities.

Sect. I. Notation of *Fractional Quantities.*

Fractional Quantities are express'd or set down like Vulgar Fractions in common Arithmetick.

$$\text{Thus} \left\{ \frac{a}{b} \quad \frac{2bc}{d} \quad \frac{5b-4a}{4d+7b} \right. \quad \begin{matrix} \textit{Numerators.} \\ \textit{Denominators.} \end{matrix}$$

How

How they come to be so, see Case 4. in the last Chapter of Division. These Fractional Quantities are managed in all respects like Vulgar Fractions in common Arithmetick.

Sect. 2. *To* Alter *or* Change *different* Fractions *into one Denomination, retaining the same Value.*

Rule.

Multiply all the Denominators into each other for a new Denominator: and each Numerator into all the Denominators but its own, for new Numerators.

Examples.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one Denomination.

First, $a \times c$, and $d \times b$, will be the Numerators, and $b \times c$, will be the common Denominator, *viz.* $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required. That is, $\frac{ca}{bc} = \frac{a}{b}$ and $\frac{bd}{bc} = \frac{d}{c}$

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought in one Denomination.

And they will be $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$ and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$ &c.

Sect. 3. *To* Bring *whole* Quantities *into* Fractions *of a given Denomination.*

Rule.

Multiply the whole Quantities into the given Denominator, for a Numerator: under which subscribe the given Denominator, and you will have the Fraction required.

Examples.

Let it be required to bring $a+b$ into a Fraction, whose Denominator is $d-a$. First $a+b \times d-a$ is $da+bd-aa-ba$.

Then $\frac{da+bd-aa-ba}{d-a}$ is the Fraction required.

Again $b+\frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d}-a$ will be $\frac{aa-da}{d}$. Also $a+b+\frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$

When

When whole Quantities are to be ſet down Fraction-wiſe, ſubſcribe an Unit for the Denominator.

Thus ab, is $\frac{ab}{1}$ And $aa - bb$, is $\frac{aa - bb}{1}$ &c.

Sect. 4. To Abbreviate, or Reduce *Fractional Quantities* into their loweſt *Denomination.*

Rule.

Divide both the Numerator and Denominator by their greateſt common Diviſor, viz. *by ſuch Quantities as are found in both; and their Quotients will be the Fraction in its loweſt Term.*

Thus $\frac{aac}{dc}$, is $\frac{aa}{d}$. And $\frac{abbb}{abc}$, is $\frac{bb}{c}$. Alſo $a + \frac{bdc}{bc} = a + d$.

In ſuch ſingle Fractions as theſe, the common Diviſors (if there be any) are eaſily diſcover'd by Inſpection only; but in compound Fractions it often proves very troubleſom, and muſt be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: Or elſe finding their common Meaſure; by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and ſo on as in Vulgar Fractions (*Sect.* 4. *Page* 51.)

Examples.

Suppoſe $\frac{aac - aad}{cd - dd}$ were to be reduced lower.

$$\begin{array}{lll} \text{Then } cd - dd) & aac - aad & \left(\frac{aa}{d}\right. \text{ the Fraction required.} \\ & \underline{aac - aad} & \\ & 0 \quad\; 0 & \end{array}$$

In this Example it ſo happens that the Numerator is divided juſt off by the Denominator; but in the next 'tis otherwiſe, and requires a double Diviſion to find out the common Meaſure. *Viz.* Let it be required to reduce $\frac{aaa - abb}{aa + 2ab + bb}$ to its loweſt Terms.

$$\begin{array}{lll} \text{Firſt } aa + 2ab + bb) & aaa - abb \;\; (a & \\ & \underline{aaa + 2aab + abb} & \\ & \quad - 2aab - 2abb & \text{the Remainder.} \\ \text{Then } - 2aab - 2abb) & aa + 2ab + bb & \left(- \frac{1}{2b} - \frac{1}{2a}\right. \\ & \underline{aa + ab} & \\ & \quad ab + bb & \\ & \quad \underline{ab + bb} & \\ & \quad 0 \quad\; 0 & \end{array}$$

Hence

Hence it appears that $-2aab-2abb$ is the common Measure; by which $aaa-abb$ being divided.

$$\text{Viz.}\ -2aab-2abb)\ \begin{array}{l} aaa-abb \\ \underline{aaa+aab} \\ \quad -aab-abb \\ \quad \underline{-aab-abb} \\ \qquad 0 \qquad 0 \end{array}\ \left(-\frac{a}{2b}+\frac{1}{2}\right.$$

Then $-\frac{a}{2b}+\frac{1}{2}$ is the new Numerator.

And ☞ $-\frac{1}{2b}-\frac{1}{2a}$ is the new Denominator.

But $-\frac{a}{2b}+\frac{1}{2}=-\frac{2a+2b}{4b}=-\frac{a+b}{2b}$ the Numerator.

And $-\frac{1}{2b}-\frac{1}{2a}=-\frac{2a-2b}{4ba}=-\frac{a-b}{2ba}$ the Denominator.

Let both be multiplied with $2ba$, and you will have

$\frac{-aa+ab}{-a-b}$ the Numerator. the Denominator. Or changing the Signs of all the Quantities, and it will be $\frac{aa-ab}{a+b}$ the new Fraction required.

That is, $\frac{aa-ab}{a+b}=\frac{aaa-abb}{aa+2ab+bb}$.

Again let it be required to reduce $\frac{dd-bb}{ddd-bbb}$,

The common Measure of this Fraction will be the easiest found (as appears from Trials) by dividing the Denominator by the Numerator, &c.

$$\text{Thus}\ dd-bb)\ \begin{array}{l} ddd-bbb\ (d \\ \underline{ddd-bbd} \\ \quad +bbd-bbb)\ dd-bb\left(\frac{d}{bb}\right. \\ \qquad\qquad\qquad \underline{dd-bd} \\ \qquad\qquad +bd-bb)\ bbd-b^3\ (b \\ \qquad\qquad\qquad\qquad \underline{bbd-b^3} \\ \qquad\qquad\qquad\qquad 0 \qquad 0 \end{array}$$

Hence it appears, that $bd-bb$ is the common Measure that will divide both the Numerator and the Denominator.

Conse-

Consequently
$$\begin{array}{l} bd - bb\,)\; dd - bb \;\left(\dfrac{d}{b} + 1\right. \text{ is the new Numerator.} \\ \qquad\qquad\;\; dd - db \\ \hline \qquad\qquad\quad + db - bb \\ \qquad\qquad\qquad db - bb \\ \hline \qquad\qquad\qquad 0 \qquad 0 \end{array}$$

And
$$\begin{array}{l} bd - bb\,)\; ddd - bbb \;\left(\dfrac{dd}{b} + d + b\right. \text{ the new Denominator.} \\ \qquad\qquad\;\; ddd - ddb \\ \hline \qquad\qquad\quad + ddb - bbb \\ \qquad\qquad\qquad ddb - bbd \\ \hline \qquad\qquad\qquad\quad + bbd - bbb \\ \qquad\qquad\qquad\qquad bbd - bbb \\ \hline \qquad\qquad\qquad\qquad 0 \qquad 0 \end{array}$$

Let both be multiplied with b, and then you will have $\left\{\dfrac{d + b}{dd + bd + bb}\right.$ $\begin{array}{l}\text{the Numerator} \\ \text{the Denominator}\end{array}\Big\}$ of the Fraction required.

But if after all Means used (as above) there cannot be found one common Measure to both the Numerator and Denominator, then is that Fraction in its least Terms already.

Note, *These Operations will be well understood by a Learner after he hath past through Multiplication, and Division of Fractions.*

Sect. 5. Addition and Subtraction of *Fractional* Quantities.

The given Fractions being of one Denomination, or if they are not, make them so, *per Sect.* 4. Then

Rule.

Add or subtract their Numerators, as Occasion requires, and to their Sum, or Difference, subscribe the common Denominator: As in Vulgar Fractions.

Examples in Addition.

	1	$\dfrac{bb}{c}$	$\dfrac{a+b}{d}$	$\dfrac{2a-b}{d+c}$	$\dfrac{a-b+d}{d+a}$
	2	$\dfrac{aa}{c}$	$\dfrac{2a+c}{d}$	$\dfrac{2b-a}{d+c}$	$\dfrac{a+b-d}{d+a}$
1 + 2	3	$\dfrac{bb+aa}{c}$	$\dfrac{3a+b+c}{d}$	$\dfrac{a+b}{d+c}$	$\dfrac{2a}{d+a}$

Examples in Subtraction.

	1	$\frac{bb+aa}{c}$	$\frac{a+b}{d+c}$	$\frac{3a+b+c}{d}$	$\frac{2b}{d+a}$
	2	$\frac{bb}{c}$	$\frac{2b-a}{d+c}$	$\frac{2a+c}{d}$	$\frac{a+b-d}{d+a}$
1 − 2	3	$\frac{aa}{c}$	$\frac{2a-b}{d+c}$	$\frac{a+b}{d}$	$\frac{-a+b+d}{d+a}$

Sect. 6. Multiplication of *Fractional Quantities.*

First, prepare *mix'd Quantities* (if there be any) by making them *improper Fractions*, and *whole Quantities* by subscribing an *Unit* under them. As *per Sect.* 3. Then

Rule.

Multiply the Numerators together for a new Numerator: And the Denominators together for a new Denominator. As in Vulgar Fractions.

Thus			
	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1 × 2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

Suppose it were required to multiply $2a+\frac{b}{c}-25$ with $3b+4c$. These prepared for the Work (*per Sect.* 3.) will stand

Thus {	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
1 × 2	3	$\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$
Or 3	4	$6ba-71b+8ac-100c+\frac{3bb}{c}$ *per Sect.* 4.

N. B.

N. B. Any Fraction is multiplied with its Denominator by casting off, or taking the Denominator away.

Thus $\frac{b}{a} \times a$ gives b. For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$ &c.

Sect. 7. Division of *Fractional Quantities.*

The Fractional Quantities being prepar'd, as directed in the last *Section.* Then

Rule.

Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator. As in Vulgar Fractions.

Examples.

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$ the Work may stand

Thus $\frac{ab}{c}\Big)\ \frac{abd}{cf}\ \Big(\frac{abdc}{abcf} = \frac{d}{f}$ per *Sect.* 4.

Or thus				
	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa-ab+bb}{c}$
$1 \div 2$	3	$\frac{d}{f}$	$\frac{aa+ba}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppose it were required to divide $aa + \frac{3abb}{a+4b}$

By $a+b$. Then the Work prepared will stand

Thus $\frac{a+b}{1}\Big)\ \frac{aaa+4aab+3aabb}{a+4b}\ \Big(\frac{aaa+4aab+3abb}{aa+5ba+4bb}$

But $\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3ba}{a+4b}$ (*per Sect* 4.)

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators.

Thus, if $\frac{ab^3}{c}$ were to be divided by $\frac{bb}{c}$ it will be

$bb)\ ab^3\ (\ ab$ the Quotient required.

 For

For $\frac{bb}{c}\Big)\frac{ab^3}{c}\Big(\frac{ab^3c}{bbc}$ But $\frac{ab^3c}{bbc} = ab$ (*per Sect.* 4.)

Again, Suppose it were required to divide $\frac{a^3 - abb}{c - d}$

By $\frac{aa + 2ab + bb}{c - d}$ Casting off $c - d$ in both, it will be

$$aa + 2ab + bb\Big)\,aaa - abb\quad\Big(\frac{aa - ba}{a + b}\ \&c.$$

Sect. 8. Involution of *Fractional Quantities.*

Rule.

Involve the Numerator into it self, for a new Numerator; and the Denominator into it self, for a new Denominator; each as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b + d}{a - c}$
1 ☉ 2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb + 2bd + dd}{aa - 2ac + cc}$
1 ☉ 3	3	$\frac{bbb}{aaa}$	$\frac{27bbbccc}{8aaaddd}$	$\frac{bbb + 3bbd + 3bdd + ddd}{aaa - 3aac + 3acc - ccc}$

Sect. 9. Evolution of *Fractional Quantities.*

If the Numerator and Denominator of the given Fraction have each of them such a Root as is required; (which very rarely happens.) Then evolve them; and their respective Roots will be the Numerator, and Denominator of the new Fraction required.

Thus	$\frac{9aabb}{4dd}$	$\frac{aa + 2ab + bb}{aa - 2ab + bb}$
1 ω 2	$\frac{3ab}{2d}$	$\frac{a + b}{a - b}$

Again	1	$\frac{27aaabbb}{8ddd}$	$\frac{aaa + 3aab + 3abb + bbb}{aaa - 3aab + 3abb - bbb}$
1 ω 3	2	$\frac{3ab}{2d}$	$\frac{a + b}{a - b}$

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; or the Deno-

Denominator may have such a Root, when the Numerator hath not. In those Cases the Operations may be set down

Thus	1	$\frac{aabb}{ddd}$	$\frac{aaa+4bb-dd}{aa+2ab+bb}$
1 ω 2	2	$\frac{ab}{\sqrt{ddd}}$	$\frac{\sqrt{aaa+4bb-dd}}{a+b}$

But when neither the Numerator, nor the Denominator have just such a Root as is required; prefix the Radical Sign of the Root to the Fraction; and then it becomes a Surd, as in the last Step, which brings me to the Business of managing Surds.

CHAP. IV.

Of Surd Quantities.

The whole Doctrine of Surds (as they call it) were it fully handled, would require a very large Explanation (to render it but tolerably intelligible) even enough to fill a Treatise it self; if all the various Examples that may be of Use to make it easy should be inserted; without which 'tis very intricate and troublesom for a Learner to understand. But now those tedious Reductions of Surds, which were heretofore thought useful to fit Æquations for such a Solution, as was then understood, are wholly laid aside as useless: Since the new Methods of resolving all Sorts of Æquations renders their Solution equally easy, although their Powers are never so high.

Nay, ever since the true Use of Decimal Arithmetick hath been well understood, the Business of Surd Numbers has been managed that Way; as appears by several Instances of that kind, in Dr. *Wallis's History* of *Algebra*. from *Page* 23 to 29.

I shall therefore, for Brevity's sake, pass over those tedious Reductions, and only shew the young Algebraist how to deal with such Surd Quantities as may arise in the Solution of hard Questions.

Sect. 1. Addition and Subtraction of *Surd Quantities*.

Case 1. When the Surd Quantities are Homogeneal (*viz.* they and their Indices are alike) Add, or Subtract the

the Rational Part, if are joined to any, and to their Sum, or Difference, adjoin the Irrational or Surd.

Examples in Addition.

	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1 + 2	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

	1	$4d\sqrt[3]{aa}$	$b+\sqrt[3]{aa+cc}$	$bc\sqrt[5]{aa+d}$
	2	$d\sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc\sqrt[5]{aa+d}$
1 + 2	3	$5d\sqrt[3]{aa}$	$b+c$	$4bc\sqrt[5]{aa+d}$

Examples in Subtraction.

	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1 + 2	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

	1	$5d\sqrt[3]{aa}$	$b+c$	$4bc\sqrt[5]{aa+d}$
	2	$4d\sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc\sqrt[5]{aa+d}$
1 − 2	3	$d\sqrt[3]{aa}$	$b+\sqrt[3]{aa-cc}$	$bc\sqrt[5]{aa+d}$

Case 2. When the Surd Quantities are Heterogeneal (*viz.* they or their Indices are unlike) they are only to be added, or ſubtracted by their Signs, *viz.* + or − And from thence will ariſe Surds, either Binomial, or Reſidual.

Examples in Addition.

	1	$\sqrt[3]{ba}$	$4d\sqrt{a}$	$\sqrt[2]{ac-ba}$
	2	$\sqrt{ba}$	$3b\sqrt{ac}$	$\sqrt[3]{ac+ba}$
1+2	3	$\sqrt[3]{ba}+\sqrt{ba}$	$4d\sqrt{a}:+3b\sqrt{ac}$	$\sqrt[2]{ac-ba}:+\sqrt[3]{ac+ba}$

Examples in Subtraction.

	1	$\sqrt{bc}$	$b-d\sqrt{aaa+ca}$
	2	$\sqrt{ba}$	$d-2a\sqrt{bd+dd}$
1 − 2	3	$\sqrt{bc}:-\sqrt{ba}$	$b-d\sqrt{aaa+ca}:-d+2a\sqrt{bd+dd}$

Sect. 2. Multiplication of *Surd Quantities.*

Case 1. When the Quantities are Pure Surds of the same kind; Multiply them together, and to their Product prefix their radical Sign.

Examples.

	1	$\sqrt{b}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$
	2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
1 × 2	3	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$

Case 2. If the Surd Quantities of the same kind (as before) are joined to rational Quantities, then multiply the Rational into the Rational; and the Surd into the Surd, and join their Products together.

Examples.

	1	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
1 × 2	3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$

Sect. 3. Division of *Surd Quantities.*

Case 1. When the Quantities are pure Surds of the same kind, and can be divided off (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

Examples.

	1	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$
	2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
1 ÷ 2	3	$\sqrt{a}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$

Case 2. If Surd Quantities of the same kind, are joined to Rational Quantities; then divide the Rational by the Rational, if it can be, and to their Quotient, join the Quotient of the Surd, divided by the Surd, with its first radical Sign.

Examples.

	1	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
1 ÷ 2	3	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$

Note

Note, If any Square be divided by its Root, the Quotient will be its Root.

Examples.

	1	a	$bb+2bc+cc$	$daaa-2bbaa+bbbb$
	2	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$
$1 \div 2$	3	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$

Sect. 4. Involution of *Surd Quantities.*

Case 1. When the Surds are not joined to Rational Quantities; they are involved to the same Height as their Index denotes, by only taking away their Radical Sign.

Examples.

	1	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa-bb}$	$\sqrt{5a-da}$
1 ⊕ 2	2	a	bca	$aa-bb$	$5a-da$

Case 2. When the Surds are joined to Rational Quantities; Involve the Rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their Radical Sign is taken away. As before.

Examples.

	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa-dd}$
1 ⊕ 2	bba	$25ddca$	$9bbaa-9bbdd$

	1	$a\sqrt[3]{bc}$	$3d\sqrt[3]{aa+bb}$	$da\sqrt[3]{b}$
1 ⊕ 3	2	$aaabc$	$27dddaa+27dddbb$	$dddaaab$

The Reason of only taking away the Radical Sign, as in Case 1. is easily conceived, if you consider that any Root, being involved into it self, produces a Square, *&c.*

And from thence the Reason of those Operations performed by the second Case may be thus stated.

Suppose $b\sqrt{a}=x$. Then $\sqrt{a}=\frac{x}{b}$ *per Axiom* 4. and both Sides of the Æquation being equally involved, it will be $a=\frac{xx}{bb}$. Then multiplying both Sides of the Æquation into bb, it will become $bba=xx$. *per Axiom* 3.

Which was to be proved.

Again

Again, let $5d\sqrt{ca}=x$. Then $\sqrt{ca}=\frac{x}{5d}$

And $ca=\frac{xx}{25dd}$ Consequently $25ddca=xx$

Alſo from hence it will be eaſy to deduce the Reaſon of multiplying Surd Quantities, according to both the Caſes. For

Suppoſe	1	$\sqrt{b}=z$	Example 1. Caſe 1.
	2	$\sqrt{a}=x$	
1 Ⓖ 2	3	$b=zz$	
2 Ⓖ 2	4	$a=xx$	
3 × 4	5	$ba=zzxx$.	*per Axiom* 3.
5 ɯ 2	6	$\sqrt{ba}=zx$.	which was to be proved.

Let	1	$d\sqrt{bc}=z$	Example 1. Caſe 2.
	2	$3b\sqrt{a}=x$	
1 ÷ d	3	$\sqrt{bc}=\frac{z}{d}$	
2 ÷ $3b$	4	$\sqrt{a}=\frac{x}{3b}$	
4 × 3	5	$\sqrt{bca}=\frac{zx}{3bd}$.	from what is proved above
5 × $3bd$	6	$3bd\sqrt{bca}=zx$, &c.	for the reſt.

Diviſion being the Converſe to Multiplication, needs no other Proof.

CHAP. V.

Concerning the Nature of Equations, *and how to prepare them for a* Solution.

When any Problem or Queſtion is propoſed to be analytically reſolved; it is very requiſite that the true Deſign or Meaning thereof, be fully and clearly comprehended (in all its Parts) that ſo it may be truly abſtracted from ſuch ambiguous Words as Queſtions of this kind are often diſguiſed with; otherwiſe it will be very difficult, if not impoſſible, to ſtate the Queſtion right in its ſubſtituted Letters, and ever to bring it to an Æquation, by ſuch various Methods of ordering thoſe Letters as the Nature of the Queſtions may require.

Now

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of ſuch leading Queſtions as are in themſelves very eaſy.

And for that Reaſon I have inſerted a Collection of ſeveral Queſtions; wherein there is great Variety.

Having got ſo clear an Underſtanding of the Queſtion propoſed, as to place down all the Quantities concerned in their due Order, *viz.* all the ſubſtituted Letters, in ſuch Order as the Nature of the Queſtion requires; the next thing muſt be to conſider whether it be limited or not. That is, whether it admits of more Anſwers than one. And to diſcover that, obſerve the two following Rules.

Rule.

When the Number of the Quantities ſought, exceed the Number of the given Æquations, the Queſtion is capable of innumerable Anſwers.

Example.

Suppoſe a Queſtion were propoſed thus; There are three ſuch Numbers, that if the firſt be added to the ſecond, their Sum will be 22. And if the ſecond be added to the third, their Sum will be 46. What are thoſe Numbers?

Let the three Numbers be repreſented by three Letters, thus, call the firſt a, the ſecond e, and the third y.

$$\text{Then} \left\{ \begin{array}{l} a+e=22 \\ e+y=46 \end{array} \right\} \text{according to the Queſtion.}$$

Here the Number of Quantities ſought are three; *viz.* a, e, y, and the Number of the given Æquations are but two. Therefore this Queſtion is not limited, but admits of various Anſwers; becauſe for any one of thoſe three Letters, you may take any Number at Pleaſure, that is leſs than 22. Which with a little Conſideration will be very eaſy to conceive.

Rule 2.

When the Number of the given Æquations (not depending upon one another) are juſt as many as the Number of the Quantities ſought; then is the Queſtion truly limited, viz. *each Quantity ſought hath but one ſingle Value.*

As for Inſtance, Let the aforeſaid Queſtion be propoſed thus. There are three Numbers (a, e, and y, As before) if the firſt be added to the ſecond, their Sum will be 22;

if

if the Second be added to the Third, their Sum will be 46; and if the First be added to the Third, their Sum will be 36. What are the Numbers?

That is, $a+e=22$. $e+y=46$. and $a+y=36$.

Now the Queſtion is perfectly limited, each ſingle Quantity having but one ſingle Value, to wit $a=6$, $e=16$, and $y=30$.

N. B. If the Number of the given Æquations exceeds the Number of the Quantities ſought; they not only *limit* the Queſtion, but oftentimes *render* it impoſſible, by being propos'd inconſiſtent one to another.

Having truly ſtated the Queſtion in its ſubſtituted Letters, and found it limited to one Anſwer, (or at leaſt ſo bounded as to have a certain determinate Number of Anſwers) then let all thoſe ſubſtituted Letters be ſo ordered or compared together, either by Adding, Subtracting, Multiplying, or Dividing them, *&c.* according as the Nature of the Queſtion requires, until all the unknown Quantities, except One, are caſt off or vaniſhed; but therein great Care muſt be taken to keep them to an exact Equality; and when that unknown Quantity, or ſome Power of it (as *Square*, *Cube*, &c.) is found Equal to thoſe that are known; then the Queſtion is ſaid to be brought to an Equation, and conſequently to a *Solution*, *viz.* fitted for an Anſwer.

But no particular Rules can be preſcribed for the caſting off, or getting away Quantities out of an Æquation; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens ſo, that the unknown Quantity which is retained in the Æquation, is ſo mix'd and entangled with thoſe that are known; that it often requires ſome Trouble and Skill to bring it (or its Powers, *&c.*) to one Side of the Æquation, and thoſe that are known to the other Side; (ſtill keeping them to a juſt Equality) which the Ingenious *VAN SCHOTEN* in his *Principia Matheſeos Univerſalis*, calls *Reduction* of *Æquations*.

The Buſineſs of Reducing Æquations (as of moſt, if not all *Algebraic* Operations) is grounded and depends upon a right Application of the five *Axioms* propoſed in Page 146. and therefore, if thoſe *Axioms* be well underſtood, the Reaſon of ſuch Operations muſt needs appear very plain, and the Work be eaſily performed; as in the following *Sections*.

Sect. 1. *Of* Reduction *by* Addition.

Reduction by *Addition* is grounded upon *Axiom* 1. and is only the transposing (*viz.* the removing) of any negative Quantity from either Side of an Æquation to the other Side, with the Sign + before it; as in these.

Examples.

Suppose	1	$a-b=d$
Then	2	$a=d+b$
For	3	$b=b$
$1+3$	4	$a=d+b$

Again,

Let	1	$aa-d=c-aa$
$1+d$	2	$aa=c-aa+d$
$2+aa$	3	$2aa=c+d$

Let	1	$3a-4=6-a$
$1+\bar{4}$	2	$3a=6+4-a$
$2+a$	3	$4a=6+4=10$

Note, *When any absolute Number is register'd in the Margin, you must draw a Line over it, to distinguish it from the other Numbers. As* $\bar{4}$ *in the 2d Step of this Example.*

Let	1	$aa-dc-b=dd-2ba$
$1+b$	2	$aa-dc=dd-2ba+b$
$2+dc$	3	$aa=dd-2ba+b+dc$
$3+2ba$	4	$aa+2ba=dd+b+dc$

Suppose	1	$2da-d=cc-3baa-aaa$
$1+aaa$	2	$aaa+2da-d=cc-3baa$
$2+3baa$	3	$aaa+3baa+2da-d=cc$
$3+d$	4	$aaa+3baa+2da=cc+d$. &c.

Sect. 2. *Of* Reduction *by* Subtraction.

Reduction by *Subtraction* is grounded upon *Axiom* 2. and is perform'd by transposing (or removing) any affirmative Quantity from either Side of the Æquation, to the other Side, with the Sign — before it. As in these

Examples.

Suppose	1	$a+b=d$
And	2	$b=b$
$1-2$	3	$a=d-b$

Let	1	$3a+4=6+a$
$1-a$	2	$2a+4=6$
$2-\bar{4}$	3	$2a=6-4=2$

Suppose	1	$aa+dc+b=dd+2ba$
$1-2ba$	2	$aa-2ba+dc+b=dd$
$2-dc$	3	$aa-2ba+b=dd-dc$
$3-b$	4	$aa-2ba=dd-dc-b$

Let	1	$aaa+d=cc+3baa+2da$
$1-3baa$	2	$aaa-3baa+d=cc+2da$
$2-2da$	3	$aaa-3baa-2da+d=cc$
$3-d$	4	$aaa-3baa-2da=cc-d$

Sect. 3. *Of* Reduction *by* Multiplication.

Fractional Quantities in any Æquation, are brought into whole Quantities; by multiplying every Term in the Æquation with the Denominators of the Fractions, *per* Axiom 3. As in these

Examples.

Suppose	1	$\frac{a}{5}=6$
Then	2	$a=6\times5=30$. For $\frac{a}{5}\times5=\frac{5a}{5}=a$

Let	1	$3a=\frac{dc}{2b}$	Suppose	1	$a=\frac{dd}{a-b}$
$1\times2b$	2	$6ba=dc$	$1\times a-b$	2	$aa-ba=dd$

Suppose	1	$\frac{aa}{b}+c+f=\frac{dx}{a}$
$1\times b$	2	$aa+bc+bf=\frac{dxb}{a}$
$2\times a$	3	$aaa+bca+bfa=dxb$

Suppose	1	$\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$
$1\times aa-bb$	2	$aaa=\frac{baaa-bbaa-bbba+bbbb}{a+b}$
$1\times a+b$	3	$aaaa+baaa=baaa-bbaa-bbba+bbbb$

Sect. 4. *Of* Reduction *by* Division.

When any Quantity (either known, or unknown) is in every Term of an Æquation; if the whole Æquation be divided by that Quantity, it will be reduc'd into lower Terms, *per* Axiom 4. As in these

Examples.

Suppose	1	$baa+bca=bcd$	Let	1	$aa=7a$
$1\div b$	2	$aa+ca=cd$	$1\div 1a$	2	$a=7$

Let	1	$ffaa+ffcaa-ffa=ffda+ffdda$
$1\div ff$	2	$aa+caa-a=da+dda$
$2\div a$	3	$a+ca-1=d+dd$

Or when the unknown Quantity is multiplied (*viz.* join'd) with any that is known; let the whole Æquation be divided by the known Quantity, that so the unknown may be cleared.

As in these

Examples.

Suppose	1	$ba-ca=d$	Let	1	$caa-daa=cd-dd$
$1\div b-c$	2	$a=\frac{d}{b-c}$	$1\div c-d$	2	$aa=\frac{cd-dd}{c-d}=d$

Suppose	1	$bbaaa-2bbaa=bda+cba$
$1\div ba$	2	$baa-2ba=d+c$
$2\div b$	3	$aa-2a=\frac{d+c}{b}$

Let	1	$49daa+42aa=7bca+21ca$
$1\div 7$	2	$7daa+6aa=bca+3ca$
$2\div a$	3	$7da+6a=bc+3c$
$3\div a$	4	$a=\frac{bc+3c}{7d+6}$

Sect. 5. *Of* Reduction *by* Involution.

When there happens to be an Æquation, between any Homogeneal or like Surds; take away the radical Signs from the Quantities, and they will become Rational.

As in these

Examples.

Suppose	1	$\sqrt{a}=\sqrt{d+c}$	Let	1	$\sqrt[3]{aa}=\sqrt[3]{db+bc}$	*per Sect.* 4. *Ch.* 3.
1 ⊛ 2	2	$a=d+c$	1 ⊛ 3	2	$aa=db+bc$	

Or if one Side of the Æquation consists of Surd Quantities, and the other Side be rational; then involve the rational Quantities

Quantities to the ſame Power (or Height) with the Index of the Surd, and take away the Radical Sign.

As in theſe

Examples.

Let	1	$\sqrt{a}=6$	Suppoſe	1	$\sqrt{a}=b+c$
1 ⊙ 2	2	$a=36$	1 ⊙ 2	2	$a=bb+2bc+cc.$
Suppoſe	1	$\sqrt[3]{aa-ba}=d$	Let	1	$\sqrt[5]{aa}=7$
1 ⊙ 3	2	$aa-ba=ddd$	1 ⊙ 5	2	$aa=16807.$

Sect. 6. *Of* Reduction *by* Evolution.

When any ſingle Powers of the unknown Quantity is on one Side of an Æquation; Evolve both Sides of the Æquation, according as the Index of that Power denotes, and their Roots will be equal. As in theſe

Examples.

Suppoſe	1	$aa=36$	Let	1	$aaa=27$
1 ω 2	2	$a=\sqrt{36}=6$	1 ω 3	2	$a=\sqrt[3]{27}=3$, &c.
Suppoſe	1	$aa=bb-dd$	Let		$aaa=b^3+3bbc+3bcc+c^3$
1 ω 2	2	$a=\sqrt{bb-dd}$	1 ω 3		$a=b+c$

Or if any compound Power of the unknown Quantity be at one Side of the Æquation (that hath a true Root of its Kind) Evolve both Sides of the Æquation, and it will be depreſs'd into lower Terms. As in theſe

Examples.

Suppoſe	1	$aa+2ba+bb=dd$	$aa-2ba+bb=ddcc$
1 ω 2	2	$a+b=d$	$a-b=dc.$

Here follow a few Examples of Æquations, wherein all the foregoing Reductions are promiſcuouſly uſed.

As Occaſion requires.

Example 1.

Suppoſe	1	$\frac{aa+c-d}{4}=\frac{g-aa}{b}$	What is $a=$ to
1 × 4	2	$aa+c-d=\frac{4g-4aa}{b}$	

2 × b	3	$baa+bc-bd=4g-4aa$
3 + $4aa$	4	$baa+4aa+bc-bd=4g$
4 + bd	5	$baa+4aa+bc=4g+bd$
5 − bc	6	$baa+4aa=4g+bd-bc$
6 ÷ $b+4$	7	$aa=\frac{4g+bd-bc}{b+4}$
7 ω 2	8	$a=\sqrt{\frac{4g+bd-bc}{b+4}}$. As was required.

Example 2.

Suppoſe	1	$\frac{a+354}{a}=\frac{3a}{354-a}$ What is the Value of a.
1 × a	2	$a+354=\frac{3aa}{354-a}$
2 × $354-a$	3	$125316-aa=3aa$
3 + aa	4	$4aa=125316$
4 ÷ 4	5	$aa=31329$
5 ω 2	6	$a=\sqrt{31329}=177$ the Value of a requir'd.

Example 3.

Suppoſe	1	$\sqrt{\frac{aa+3bb}{4}}-\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{baa}{c}}:a=?$
1 ⊕ 2	2	$\frac{aa+3bb}{4}-2\sqrt{\frac{aa+3bb}{4}}\times\sqrt{\frac{aa-3bb}{4}}+\frac{aa-3bb}{4}=\frac{baa}{c}$
That is	3	$\frac{aa}{2}-\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}$
For		$\frac{aa+3bb}{4}+\frac{aa-3bb}{4}=\frac{2aa}{4}=\frac{aa}{2}$
And		$2\sqrt{\frac{aa+3bb}{4}}=\sqrt{\frac{4aa+12bb}{4}}=\sqrt{aa+3bb}$
Then		$\sqrt{aa+3bb}\times\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{a^4-9b^4}{4}}$
3 + √ &c.	4	$\frac{aa}{2}=\frac{baa}{c}+\sqrt{\frac{a^4-9b^4}{4}}$

$4 - \frac{baa}{c}$	5	$\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$
$5 \circledast 2$	6	$\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$
$6 + \frac{ba^4}{c}$	7	$\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b4}{4} + \frac{ba^4}{c}$
$7 \pm$	8	$\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$
$8 \div b$	9	$\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
$9 \times cc$	10	$ba^4 + \frac{9ccb^3}{4} = ca^4$
10×4	11	$4ba^4 + 9ccb^3 = 4ca^4$
$11 - 4ba^4$	12	$9ccb^3 = 4ca^4 - 4ba^4$
$12 \div$		$aaaa = \frac{9ccb^3}{4c - 4b}$
For	13	$4c - 4b \times a^4 = 4ca^4 - 4ba^4$
13 ɯ 2	14	$aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$
14 ɯ 2	15	$a = \sqrt{:\sqrt{\frac{9ccb^3}{4c - 4b}}}$. As was required.

By help of these Reductions (properly applied) the unknown Quantity, (*a*) or its Powers, are cleared and brought to one Side of an Æquation; and if the unknown Quantity (*a*) chance to be equal to those that are known, the Question is answered.

As in the first Example of *Sect.* 1. and 2.

Or if any single Power of the unknown Quantity (*a*) is found equal to those that are known, then the respective Root of the known Quantities, is the Answer; as in the first four *Examples* of *Sect.* 6, *&c.*

But when the Powers of the unknown Quantity are either mixed with their Root; As $aa + ba = dd$, *&c.* Or do consist of different Powers; As $aaa + baa = dd$, *&c.* Then they are called affected, or adfected Æquations, which require other Methods to resolve them, *viz.* to find out the Value of (*a*) as shall be shewed further on.

CHAP.

CHAP. VI

Of Proportional Quantities; *both* Arithmetical, Geometrical *and* Musical.

What hath been ſaid of Numbers in Arithmetical Progreſſion, *Chap. 6. Part* 1. may be eaſily applied to any Series of homogeneal or like Quantities.

Sect. 1. *Of* Quantities *in* Arithmetical Progreſſion.

Thoſe Quantities are ſaid to be in the moſt ſimple or natural Progreſſion, that begin their Series of Increaſe or Decreaſe with a Cypher.

$$\text{Thus}\begin{cases} 0 : a : 2a : 3a : 4a : 5a : 6a : \&c. & \text{Increaſing.} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : \&c. & \text{Decreaſing.} \end{cases}$$

Or Univerſally, putting a the firſt Term in the Progreſſion, and e the common Exceſs, or Difference.

$$\text{Then}\begin{cases} a : a+e : a+2e : a+3e : a+4e : a+5e : a+6e : \&c. \\ a : a-e : a-2e : a-3e : a-4e : a-5e : a-6e : \&c. \end{cases}$$

In the firſt of theſe Series it's evident, that if there be but three Terms; the Sum of the Extremes will be double to the Mean.

As in theſe, $0 : a : 2a$: Or, $a : 2a : 3a$: Or, $2a : 3a : 4a$, &c. viz. $2a+0=a+a$: Or, $a+3a=2a+2a$, &c.

Alſo, in the ſecond Series, either Increaſing, or Decreaſing, it is evident, that if the Terms be $a : a+e : a+2e$, &c. Increaſing; Then $a+a+2e$, viz. $2a+2e$ the Sum of the Extremes, is double to $a+e$ the Mean; Or if they be $a : a-e : a-2e$, &c. Decreaſing, then $a+a-2e$: viz. $2a-2e$ the Sum of the Extremes, is double to $a-e$ the Mean. And ſo it will be in any other three of the Terms.

Secondly, If there are four Terms; then the Sum of the two Extremes, will be equal to the Sum of the two Means. As in theſe $a : a+e : a+2e : a+3e$, in the Series Increaſing; Here $a+a+3e=a+e+a+2e$.

Alſo in theſe, $a : a-e : a-2e : a-3e$ in the Series Decreaſing; here $a+a-3e=a-e+a-2e$, &c. in any other four Terms.

Conſequently, If there are never ſo many Terms in the Series, the Sum of the two Extremes, will always be equal to the Sum of

of any two Means, that are equally diſtant from thoſe Extremes. As in theſe, $a : a+e : a+2e : a+3e : a+4e : a+5e$: &c. Here $a+a+5e = a+e+a+4e = a+2e+a+3e$, &c. And if the Number of Terms be odd; the Sum of the two Extremes will be double to the middle Term, *&c.* As in *Corol.* 1. *Chap.* 6. before mention'd.

CONSECTARY 1.

Whence it follows, (and is very eaſy to conceive) that if the Sum of the two Extremes be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.

Now for the eaſier reſolving ſuch Queſtions as depend upon theſe Progreſſional Quantities.

Let $\begin{cases} a = \text{the firſt Term, as before.} \\ y = \text{the laſt Term.} \\ e = \text{the Common Exceſs, \&c. as before.} \\ N = \text{the Number of all the Terms.} \\ S = \text{the Sum of all the Series, viz. of all the Terms.} \end{cases}$

Then will $a+y \times N = 2S$, by the precedent *Conſectary*: That is, $Na+Ny = 2S$. Conſequently $\frac{Na+Ny}{2} = S$, the Sum of all the Series, be the Terms never ſo many. *Thirdly*, In theſe Series, it is eaſy to perceive, that the common Difference (e) is ſo often added to the laſt Term of the Series; as are the Number of Terms, except the firſt; That is, the firſt Term (a) hath no Difference added to it, but the laſt Term hath ſo many times (e) added to it, as it is diſtant from the firſt.

Conſequently, the Difference betwixt the two Extremes, is only the common Difference (e) multiplied into the Number of all the Terms leſs Unity or 1.

That is, $N-1 \times e = y-a$, the Difference betwixt the two Extremes, *viz.* $Ne-e = y-a$.

CONSECTARY 2.

Whence it follows, that if the Difference betwixt the two Extremes be divided by the Number of Terms Leſs 1. the Quotient will be the common Difference of the Series.

To wit, $\frac{y-a}{N-1} = e$.

Now by the help of these two *Consectaries*, if any three of the aforesaid five Parts (*viz*. $a.\ y.\ e.\ N.\ S.$) be given; the other two may be easily found.

Thus,	1	$\frac{Na+Ny}{2}=S$ *As before.*
And	2	$\frac{y-a}{N-1}=e$
$2\times N-1$	3	$y-a=Ne-e$
$3+e$	4	$y-a+e=Ne$
$4\div e$	5	$\frac{y-a+e}{e}=N$ *The Number of Terms.*
1×2	6	$Na+Ny=2S$
$6-Na$	7	$Ny=2S-Na$
$7\div N$	8	$\frac{2S-Na}{N}=y$ *The last Term.*
$6-yN$	9	$Na=2S-Ny$
$9\div N$	10	$\frac{2S-Ny}{N}=a$ *The first Term.*
$6\div a+y$	11	$\frac{2S}{a+y}=N$ The *Number* of *Terms.*
5, and 11	12	$\frac{y-a+e}{e}=\frac{2S}{a+y}$ Per *Axiom* 5.
$12\times a+y$	13	$\frac{yy-aa}{e}+a+y=2S$
$13\div 2$	14	$\frac{yy-aa}{2e}+\frac{a+y}{2}=S$ The *Sum* of all the *Series.*
$14\times 2e$	15	$yy-aa+ae+ye=2Se$
$15-ae$	16	$yy-aa+ye=2Se-ae$
$16-ye$	17	$yy-aa=2Se-ae-ye$
$17\div$	18	$\frac{yy-aa}{2S-a-y}=e$ The *Common Difference.*
$3+a$	19	$Ne-e+a=y$ The *last Term.*
$19+e$	20	$Ne+a=y+e.$
$20-Ne$	21	$y+e-Ne=a$ The *first Term.*
		&c.

In like Manner you may proceed to find out any of the five Quantities ($a.\ e.\ y.\ N.\ S.$) otherways, *viz*. by varying or comparing of these Æquations one with another, you may produce new Æquations with other *Data* in them; the which I shall

I shall here omit pursuing, and leave them for the Learner's Practice.

Sect. 2. *Of* Quantities *in* Geometrical Proportion.

Geometrical Proportion continued has been already defined in *Sect.* 2. *Chap.* 6. *Part* 1. And what is there said concerning Numbers in ∺ may easily be applied to any Sort of Homogeneal Quantities that are in ∺.

The most natural and simple Series of Geometrical Proportionals, is when it begins with Unity or 1.

As $1 \,.\, a \,.\, aa \,.\, aaa \,.\, aaaa \,.\, a^5 \,.\, a^6$, &c. in ∺
For $1 : a :: a : aa :: aa : aaa :: aaa : aaaa$, &c.
Or $a \,.\, b \,.\, \frac{bb}{a} \,.\, \frac{bbb}{aa} \,.\, \frac{bbbb}{aaa} \,.\, \frac{b^5}{a^4}$. &c. are Terms in ∺
For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{b^4}{a^3} :: \frac{b^4}{a^3} : \frac{b^5}{a^4}$. &c.

That is, when all the middle Terms betwixt the two Extremes are both *Consequents* and *Antecedents*, that *Series* is in *Geometrical Proportion continued.*

Therefore in every Series of Quantities in ∺ all the Terms except the Last are *Antecedents*; and all the Terms except the First are *Consequents*.

But Universally putting a the first Term in the Series, and e the *Ratio, viz.* the common Multiplier, or Divisor, then it will be

$a \,.\, ae \,.\, aee \,.\, aeee \,.\, aeeee \,.\, ae^5 \,.\, ae^6$. &c. in ∺

Or $a \,.\, \frac{a}{e} \,.\, \frac{a}{ee} \,.\, \frac{a}{eee} \,.\, \frac{a}{eeee} \,.\, \frac{a}{e^5}$. &c. are in ∺ Decr.

For $a : ae :: ae : \frac{aaee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{aee} = \frac{a}{ee}$ $a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$ &c.

I. In any of these Series it is evident, that if three Quantities are in ∺ the Rectangle of the two Extremes will be equal to the Square of the Mean.

As in these, $a : ae : aee$ here $a \times aee = ae \times ae. = aaee$. &c.

 Or

Or $a . \frac{a}{e} . \frac{a}{ee}$ here alſo, $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$ &c.

II. If four Quantities are in ∺ the Rectangle of the Extremes will be equal to the Rectangle of the Means.

As in theſe, $a . ae . aee . aeee .$ here $a \times ae^3 = ae \times aee.$

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee}$ here alſo $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$ &c.

Conſequently, If there are never ſo many Terms in the Series of ∺ the Rectangle of the Extremes will be equal to the Rectangle of any two Means that are equally diſtant from thoſe Extremes.

As in theſe, $a . ae . aee . aeee . aeeee . ae^5$ &c.
viz. $ae^5 \times a = ae^4 \times ae.$ Or $ae^5 \times a = aeee \times aee = aae^5.$

III. If never ſo many Quantities are in ∺ it will be, As any one of the *Antecedents* is to its *Conſequent*; So is the Sum of all the *Antecedents*, to the Sum of all the *Conſequents*.

As in theſe, $\begin{cases} a . ae . aee . aeee . aeeee . ae^5, \text{ \&c. Increaſing.} \\ a \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e^5} \text{ \&c. Decreaſing.} \end{cases}$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$
viz. $a \times ae + aee + ae^3 + ae^4 + ae^5 = ae \times a + ae + aee + ae^3 + ae^4$

That is, the Rectangle of the Extremes is equal to the Rectangle of the Means; *per Second* of this *Sect.*

Note, The Ratio of any Series in ∺ increaſing is found by dividing any of the *Conſequents* by its *Antecedent*.

Thus, $a) \; ae \; (e$ Or $ae) \; aee \; (e$ &c.

But if the Series be decreaſing, then the Ratio is found by dividing any of the *Antecedents* by its *Conſequent*.

Thus, $\frac{a}{e}) \; a \; (e$ Or $\frac{a}{ee}) \; \frac{a}{e} \; (e$ &c.

CON-

CONSECTARY.

Theſe Things being premiſed, ſuch Æquations may be deduced from them, as will ſolve all ſuch Queſtions as are uſually propoſed about Quantities in Geometrical Proportion ∸ In Order to that

Let $\begin{cases} a = \text{The } \textit{First Term.} \\ e = \text{The } \textit{Common Ratio.} \\ y = \text{The } \textit{Last Term.} \\ S = \text{The Sum of all the } \textit{Terms.} \end{cases}$ (the first two as before.)

Then $S - y =$ the Sum of all the *Antecedents.*
And $S - a =$ the Sum of all the *Conſequents.*

Analogy.		
	1	$a : ae :: S - y : S - a$ *Per* the III. of this *Sect.*
1 ∵	2	$Sa - aa = aeS - aey$
$2 \div a$	3	$S - a = eS - ey$
$3 + ey$	4	$S + ey - a = eS$
$4 - S$	5	$ey - a = eS - S$
$5 \div e - 1$	6	$\frac{ye - a}{e - 1} = S$ The Sum of all the Series.
$3 \div S - y$	7	$\frac{S - a}{S - y} = e$ The *common Ratio.*
$5 + a$	8	$ey = eS + a - S$
$8 \div e$	9	$\frac{eS + a - S}{e} = y$ The laſt Term.
$4 + a$	10	$S + ey = eS + a$
$10 - Se$	11	$S + ey - eS = a$ The firſt Term.

Note, The ∵ ſet in the Margin at the ſecond Step, is inſtead of *Ergo*; and imports that the Rectangle of the two Extremes in the firſt Step, is equal to the Rectangle of the Means. And ſo for any other Proportion.

Sect. 3. *Of* Harmonical Proportion.

Harmonical or Muſical Proportion is, when of three Quantities (or rather Numbers) the Firſt hath the ſame Ratio to the Third, As the Difference between the Firſt and Second, hath to the Difference between the Second and Third. As in theſe following.

Suppoſe a, b, c in *Muſical Proportion.*

Then	1	$a : c :: b - a : c - b$
1 ∵	2	$cb - ca = ac - ba$

$2 + ca$

$2 + ca$	3	$cb = 2ac - ba$
$3 \div 2c - b$	4	$\frac{cb}{2c - b} = a$ The first Term.
$3 + ba$	5	$2ac = cb + ba$
$5 \div c + a$	6	$\frac{2ac}{c + a} = b$ The second Term.
$5 - cb$	7	$2ac - cb = ba$
$7 \div 2a - c$	8	$\frac{ba}{2a - b} = c$ The third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the Fourth, as the Difference between the First and Second hath to the Difference between the Third and Fourth. That is, Let $a.\ b.\ c.\ d$ be the four Terms, *&c.*

Then	1	$a : d :: b - a : d - c$
$1 \because$	2	$db - da = da - ca$
$2 + da$	3	$db = 2da - ca$
$3 \div 2d - c$	4	$\frac{db}{2d - c} = a$
$3 \div d$	5	$b = \frac{2da - ca}{d}$
$3 + ca$	6	$db + ca = 2da$
$6 - db$	7	$ca = 2da - db$
$7 \div a$	8	$c = \frac{2da - db}{a}$
$7 \div 2a - b$	9	$\frac{ca}{2a - b} = d$

CHAP. VII.

Of Proportion Disjunct, *and how to* turn Æquations *into* Analogies, &c.

Proportion Disjunct, or the *Rule of Three* in *Numbers* is already explain'd in *Chap.* 7. *Part* 1. And what hath been there said is applicable to all Homogeneous Quantities, *viz.* of Lines to Lines, *&c.*

Sect. 1.

If four Quantities (*viz.* either *Lines*, *Superficies*, or *Solids*) be Proportional: The Rectangle comprehended under the Extremes,

is

is equal to the Rectangle comprehended under the two Means, (16 *Euclid* 6.)

For Inſtance, Suppoſe, $a \,.\, b \,.\, c \,.\, d$. to repreſent the four Homogeneal Quantities in Proportion.

viz. $a : b :: c : d$. Then will $ad = bc$.
For ſuppoſe $b = 2a$ then will $d = 2c$
And it will be $a : 2a :: c : 2c$. Here the *Ratio* is 2.
but $a \times 2c = 2a \times c$. *viz.* $2ca = 2ac$
Or ſuppoſe $b = 3a$ then will $d = 3c$
And it will be $a : 3a :: c : 3c$. Here the *Ratio* is 3.
but $a \times 3c = 3a \times c$. *viz.* $3ac = 3ac$.

Or Univerſally putting e for the *Ratio* of the Proportion, *viz.* making $b = ae$, then will $d = ce$
And it will be $a : ae :: c : ce$
but $a \times ce = ae \times c$ *viz.* $ace = aec$.

Conſequently, $ad = bc$ which was to be proved.

Whence it follows, that if any Three of the four Proportional Quantities be given, the Fourth may be eaſily found.

Thus,

Let	1	$a : b :: c : d$. Then
1 ∵	2	$ad = bc$ as before
$2 \div d$	3	$a = \frac{bc}{d}$
$2 \div c$	4	$b = \frac{ad}{c}$
$2 \div b$	5	$c = \frac{ad}{b}$
$2 \div a$	6	$d = \frac{bc}{a}$
$2 \div bd$	7	$\frac{a}{b} = \frac{c}{d}$
Or $2 \div ac$	8	$\frac{b}{a} = \frac{d}{c}$

Note, *In this Manner* Euclid *in his 5th Book expreſſes the Ratio of Proportionals,* viz. *the Ratio of a to b is* $\frac{a}{b}$

If four Quantities are Proportional, they will alſo be Proportional in *Alternation, Inverſion, Compoſition, Diviſion, Converſion,* and *Mixtly. Euclid* 5. *Def.* 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be *in Direct Proportion*, as before.
Then	2	$a : c :: b : d$ *Alternate.* For $ad = bc$
And	3	$b : a :: d : c$ *Inverted.* For $ad = bc$
Also	4	$a + b : b :: c + d : d$ *Compounded.*
4 $\because$	5	$da + bd = bc + bd$ That is, $ad = bc$, as before.
Or	6	$a + c : c :: b + d : d$ *Alternately Compounded.*
6 $\because$	7	$ad + cd = bc + cd$ That is $ad = bc$.
Again,	8	$a - b : b :: c - d : d$ *Divided.*
8 $\because$	9	$ad - bd = bc - bd$ That is, $ad = bc$.
Or	10	$a - c : c :: b - d : d$ *Alternately Divided.*
10 $\because$	11	$ad - cd = bc - cd$ That is, $ad = bc$.
And	12	$a : b \pm a :: c : d \pm c$ *Converted*
12 $\because$	13	$ad \pm ac = bc \pm ac$ That is $ad = bc$.
Lastly	14	$a + b : a - b :: c + d : c - d$ *Mixtly.*
14 $\because$	15	$ac - ad + bc - bd = ac + ad - bc - bd$
15 $+$	16	$2bc = 2ad$; That is, $ad = bc$ As at first.

Note, What has been here done about whole Quantities in *Simple Proportion*, may be easily perform'd in *Fractional Quantities*; And *Surds*, &c.

For Instance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$ and it be required to find the fourth Term.

it will be $\frac{dd-cc}{fc}$ the Rectangle of the Means; which being divided by the first Extreme $\frac{ab}{c}$ it will become

$$\frac{ab}{c}\Big)\ \frac{dd-cc}{fc}\ \Big(\frac{ddc-ccc}{abfc} = \frac{dd-cc}{abf}$$ the fourth Term.

Or if $b : \sqrt{bd+bc} :: \sqrt{bd+bc}$: to a fourth Term.

Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the *Rectangle* of the *Means*, And $b)\ bd+bc\ (d+c$ the fourth Term.

That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$ &c.

Sect. 2. *Of* Duplicate *and* Triplicate Proportion.

The Proportions treated of in the last Section, are to be understood when Lines are compared to Lines, and Superficies to Su-

Superficies; or Solids to Solids; *viz.* when each is compared to that of its like Kind, which is only called Simple Proportion.

But when Lines are compared to Superficies, or Lines are compared to Solids, such Comparisons are distinguished from the former, by the Names of Duplicate and Triplicate, (*&c.*) Proportions; so that Simple, Duplicate and Triplicate, *&c.* Proportions are to be understood in a different Sense from Single, Double, Triple, *&c.* Proportions, which are only as 1, 2, 3, *&c.* to 1; but those of Simple, Duplicate, Triplicate, *&c.* Proportion, is that of $a \cdot aa \cdot aaa$., *&c.* to 1. Or if the Simple Proportions be that of a to b, whose Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$ according to *Euclid's* Way.

$$\left.\begin{array}{l}\text{Then } \frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb} \text{ is the Exponent of the Duplicate} \\ \text{And } \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3} \text{ is the Exponent of the Tr. Prop.} \\ \qquad\qquad\qquad \textit{\&c.}\end{array}\right\} \text{of } \frac{a}{b}$$

And if there are Three, Four, or more Quantities, in $\div$ as $1 \cdot a \cdot aa \cdot aaa : a^4 \cdot a^5$, &c. (As in the first Series *Sect.* 2. of the last Chapter.) Then, that of the First to the Third, Fourth and Fifth, *&c.* (*viz.* 1 To $aa \cdot aaa \cdot aaaa \cdot a^5$) is Duplicate, Triplicate, Quadruplicate, *&c.* of the First to the Second, (*viz.* of 1 To a;) And by Inversion, that of the Third, Fourth, Fifth, is Duplicate, Triplicate, *&c.* of that of the Second to the First (*a* To 1) *per Def.* 10. *Eucl.* 5. But the Nature of these Proportions will appear more evident, and be easier understood when they are applied to Practice, and illustrated by Geometrical Figures. Further on.

Sect. 3. *How to* turn Æquations *into* Analogies.

From the first *Section* of this *Chap.* it will be easy to conceive how to turn or dissolve Æquations into Analogies or Proportions.

For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are those four (or more) Quantities Proportional. By the 16 *Euclid* 6.

That is, if $ab = dc$. Then is $a : c :: d : b$.

Or $c : a :: b : d$ &c.

From whence there arises this general *Rule* for turning Æquations into Analogies.

Rule.

Divide either Side of the given Æquation, (if it can be done) into two ſuch Parts, or Factors, as being multiplied together, will produce that Side again; and make thoſe two Parts the two Extremes. Then divide the other Side of the Æquation (if it can be done) in the ſame Manner as the Firſt was, and let thoſe two Parts or Factors, be the two Means.

For Inſtance, Suppoſe $ab + ad = bd$.
Then $a : b :: d : b + d$. Or $b : a :: b + d : d$ &c.

Or taking ad from both Sides of the Æquations, and
It will be $ab = bd - ad$. Then $a : d :: b - a : b$.

Or, $b : d :: b - a : a$ &c.
Again, Suppoſe $aa + 2ae = 2by + yy$.
Here a and $a + 2e$ are the two Factors of the firſt Side in this Æquation; for $\overline{a + 2e} \times a = aa + 2ae$.
Again, y and $2b + y$ are the two Factors of the other Side.
Therefore, $a : y :: 2b + y : a + 2e$.
Or $2b + y : a + 2e :: a : y$ &c.
When one Side of any Æquation can be divided into two Factors, as before; and the other Side cannot be ſo divided, then make the Square Root of that Side either the two Extremes or the two Means.
For Inſtance. Suppoſe $bc + bd = da + g$.
Then $b : \sqrt{da + g} :: \sqrt{da + g} : c + d$.
Or $\sqrt{da + g} : b :: c + d : \sqrt{da + g}$. &c.

CHAP. VIII.

Of Subſtitution, and the Solution of Quadratic Æquations.

Sect. I. Of Subſtitution.

When new Quantities not concerned in the firſt ſtating of any Queſtion, are put inſtead of ſome that are engag'd in it, that is called Subſtitution.

For Inſtance, If inſtead of $\sqrt{bc - dc}$ you put z, or any other Letter.

That is, make $z = \sqrt{bc - dc}$. Or

Or, Suppoſe $aa + ba - ca + da = dc$, inſtead of $b - c + d$ put s, or any other Letter not engaged in the Queſtion,

Viz. $s = b - c + d$ Then $aa \mp sa = dc$

That is, if c be greater than $b + d$, it's $aa - sa = dc$.

But if $b + d$ be greater than c, Then it's $aa + sa = dc$.

And this Way of Subſtituting or putting of new Quantities inſtead of others, may be found very uſeful upon ſeveral Occaſions; *viz*. in order to make ſome following Operations in the Queſtion more eaſy, and perhaps much ſhorter than they would be without it, as you may obſerve in ſome Queſtions hereafter propoſed in this Tract.

And when thoſe Operations, in which the ſubſtituted Quantities were aſſiſting or uſeful, are performed according as the Nature of the Queſtion required, you may then (if there be Occaſion) bring the Original or firſt Quantities into the Æquation, in the Place (or Places) of thoſe ſubſtituted Quantities, which is called Reſtitution, as you will ſee further on.

Sect. 2. *The* Solution *of* Quadratic Equations.

When the Quantity ſought is brought to an Equality with thoſe that are known, and is on one Side of the Æquation, in no more than two different Powers whoſe Indices are double one to another, thoſe Æquations are called Quadratic Æquations Adfected; and do fall under the Conſideration of three Forms or Caſes.

$$\left.\begin{array}{l}\text{Caſe 1. } aa + 2ba = dc. \\ \text{Caſe 2. } aa - 2ba = dc. \\ \text{Caſe 3. } 2ba - aa = dc.\end{array}\right\} \text{ And } \left\{\begin{array}{l} a^4 + 2ba^2 = dc. \\ a^4 - 2ba^2 = dc. \\ 2ba^2 - a^4 = dc.\end{array}\right.$$

$$\text{Alſo } \left\{\begin{array}{l} a^6 + 2ba^3 = dc. \\ a^6 - 2ba^3 = dc. \\ 2ba^3 - a^6 = dc.\end{array}\right\} \text{ And } \left\{\begin{array}{l} a^8 + 2ba^4 = dc. \\ a^8 - 2ba^4 = dc. \\ 2ba^4 - a^8 = dc.\end{array}\right\} \text{ \&c.}$$

When there happen to be more Terms in one of theſe Kind of Æquations than two, and the higheſt Power of the unknown Quantity is multiplied into ſome known Co-efficients; you muſt reduce them by Diviſion; as in *Sect.* 4. of *Chap.* 5. and for the Fractional Quantities that may ariſe by thoſe Diviſions, Subſtitute another Quantity doubled.

For Inſtance, Let $baa + caa - ca - da = dc + cb$.

Then $aa - \dfrac{ca - da}{b + c} = \dfrac{dc + cb}{b + c}$ make $\dfrac{c - d}{b + c} = 2x$.

And if you pleaſe, for $\frac{dc+cb}{b+c}$ put z.

Then will $aa-2xa=z$ be the new Æquation, equal to the other, being now fitted for a Solution.

Now any of theſe three Forms of Æquations being thus prepared for a Solution, may be reduced to ſimple Powers by caſting off the ſecond or loweſt Term of the unknown Quantity; which is done by Subſtitution, thus always take half the known Co-efficient, and add it to (in Caſe 1.) or ſubtract it from (in Caſe 2.) its fellow Factor; and for their Sum, or Difference, ſubſtitute another Letter. As in theſe.

Let	1	$aa+2ba=dc$ Caſe 1.
Put	2	$a+b=e$
2 ⊛ 2	3	$aa+2ba+bb=ee$
$3-1$	4	$bb=ee-dc$
$4+dc$	5	$ee=bb+dc$
5 ɯ 2	6	$e=\sqrt{bb+dc}$
2 and 6	7	$a+b=\sqrt{bb+dc}$ *Per* Axiom 5.
$7-b$	8	$a=\sqrt{bb+dc}:-b$

Again.

Let	1	$aa-2ba=dc$ Caſe 2.
Put	2	$a-b=e$
2 ⊛ 2	3	$aa-2ba+bb=ee$
$3-1$	4	$bb=ee-dc$
$4+dc$	5	$ee=dc+bb$
5 ɯ 2	6	$e=\sqrt{dc+bb}$
2, 6	7	$a-b=\sqrt{dc+bb}$
$7+b$	8	$a=b+\sqrt{dc+bb}$

In Caſe 3. From half the known Co-efficient ſubtract its fellow Factor.

Thus, Let	1	$2ba-aa=dc$
Put	2	$b-a=e$
2 ⊛ 2	3	$bb-2ba+aa=ee$
$1+3$	4	$bb=dc+ee$
$4-dc$	5	$ee=bb-dc$

5 ɯ 2	6	$e = \sqrt{bb - dc}$
(2, 6	7	$b - a = \sqrt{bb - dc}$
7 + a	8	$b = a + \sqrt{bb - dc}$
8 — √, &c.	2	$a = b - \sqrt{bb - dc}$

And this Method holds good in thoſe other Æquations, wherein the higheſt Powers are a^4, a^6, a^8, &c. As for Inſtance.

Let	1	$a^6 + 2ba^3 = dc$ Caſe 1.
Put	2	$a^3 + b = e$
2 © 2	3	$a^6 + 2ba^3 + bb = ee$
3 — 1	4	$bb = ee - dc$
4 + dc	5	$ee = bb + dc$
5 ɯ 2	6	$e = \sqrt{bb + dc}$
2, 6	7	$a^3 + b = \sqrt{bb + dc}$
7 — b	8	$a^3 = \sqrt{bb + dc} : - b$
8 ɯ 3	9	$a = \sqrt[3]{: \sqrt{bb + dc} : - b}$

The ſame may be done with all the reſt, Care being taken to add, or ſubtract, according as the Caſe requires.

But all Quadratic Æquations may be more eaſily reſolved by completing the Square which is grounded upon the Conſideration of raiſing a Square from any Binomial, or Reſidual Root, (See *Sect.* 5. *Chap.* 1.

Viz. If $a + b$ be *involved* to a *Square*, it will be, $aa + 2ba + bb$
And if $a - b$ be ſo involved, it will be, $aa - 2ba + bb$
Whence it is eaſy to obſerve, that $aa + 2ba = dc$. Caſe 1.
And $aa - 2ba = dc$. Caſe 2. are imperfect Squares, wanting only bb to make them complete. And therefore it is that if half the known Co-efficient be involved to the ſecond Power, and the Square be added to both Sides of the Æquation, the unknown Side will become a complete Square.

Thus, Let	1	$aa + 2ba = dc.$	Here half the Co-efficient $2b$ is b, which being ſquared, is bb.
But	2	$bb = bb$	
1 + 2	3	$aa + 2ba + bb = dc + bb$ Caſe 1.	
3 ɯ 2	4	$a + b = \sqrt{dc + bb}$ As before.	

Again.

Again.

Let	1	$aa - 2ba = dc$ Case 2.
But	2	$bb = bb$
1 + 2	3	$aa - 2ba + bb = dc + bb$
3 ɯ 2	4	$a - b = \sqrt{dc + bb}$ &c. As before.

But in Case 3. you must change the Signs of all the Terms in the Æquation,

Thus	1	$2ba - aa = dc$ Case 3.
1 ±	2	$aa - 2ba = -dc$
Then	3	$aa - 2ba + bb = bb - dc$ &c.

And this Method of completing the Square holds true in those other Æquations.

Viz.	1	$aaaa + 2baa = dc$ Case 1.
For	2	$bb = bb$ As before.
1 + 2	3	$aaaa + 2baa + bb = dc + bb$
3 ɯ 2	4	$aa + b = \sqrt{dc + bb}$
4 − b	5	$aa = \sqrt{dc + bb} : - b$
5 ɯ 2	6	$a = \sqrt{: \sqrt{dc + bb} : - b}$ And so on for the rest.

Or let	1	$a^6 + 2baaa = dc$ As before, Case 1.
And	2	$bb = bb$
1 + 2	3	$a^6 + 2baaa + bb = dc + bb$
1 ɯ 2	4	$aaa + b = \sqrt{dc + bb}$
4 − b	5	$aaa = \sqrt{dc + bb} : - b$
5 ɯ 3	6	$a = \sqrt[3]{: \sqrt{dc + bb} : - b}$ &c.

COROLLARY.

Hence it is evident, that whatsoever Method is used in solving these (or indeed any other) Æquations, the Result will still be the same, if the Work be true; as you may observe from the Operations of this Section: For both these Methods here proposed, give the same Theorems in their respective Cases for the Value of (a).

Thus

Thus, when $aa + 2ba = dc$ Then

Theorem 1. $a = \sqrt{dc + bb} : - b$

And when $aa - 2ba = dc$ Then

Theorem 2. $a = b + \sqrt{dc + bb}$

Again, when $2ba - aa = dc$ Then

Theorem 3. $a = b - \sqrt{bb - dc}$

The *like Theorems* may be *easily* raised for the rest.

If the known Co-efficients (of the second or lowest Term) be any single Quantity, as $aa + ba = dc$, &c. Then is $\frac{1}{2} b$ its half, and $\frac{1}{4} bb$ will be the Square of that half: That is, $\frac{1}{2} b \times \frac{1}{2} b = \frac{1}{4} bb$. And then the Work will stand

Thus,	1	$aa + ba = dc$
1 C □	2	$aa + ba + \frac{1}{4} bb = dc + \frac{1}{4} bb$
2 ω 2	3	$a + \frac{1}{2} b = \sqrt{dc + \frac{1}{4} bb}$
$3 - \frac{1}{2} b$	4	$a = \sqrt{dc + \frac{1}{4} bb} : - \frac{1}{2} b$. And so for the rest.

Note, C □ placed in the Margin against the second Step, signifies that the imperfect Square $aa + ba$ in the first Step, is there compleated, *viz.* in the second Step.

Now by the help of these *Theorems*, it will be easy to calculate or find the Value of the unknown Quantity, (a) in Numbers.

Example 1.

Suppose $aa + 2ba = z$. Let $b = 16$ And $z = 4644$.

Then $a = \sqrt{z + bb} : - b$ *Per* Theorem 1.

But $z + bb = 4644 + 256 = 4900$ And $\sqrt{4900} = 70$

Consequently $a = 70 - 16$, viz. $a = 54$.

But every Adfected Æquation, hath as many Roots (or rather Values of the unknown Quantity) either real or imaginary, as are the Dimensions (*viz.* the Index) of its highest Power; and therefore the Quantity a, in this Æquation, hath another Value either affirmative or negative; which may be thus found.

The given Æquation is $aa + 32a = 4644$, and its Root $a = 54$

Let these two Æquations be made equal or Æquated to 0, *viz.* to Nothing.

Thus,

Thus, $aa + 32a - 4644 = 0$ And $a - 54 = 0$.
Then divide the given Æquation by its firſt Root, and the Quotient will ſhew the ſecond Value of a.

$$\text{Thus, } a - 54 = 0)\ aa + 32a - 4644 = 0\ (a + 86 = 0$$
$$\begin{array}{r} aa - 54a \\ \hline +86a - 4644 \\ 86a - 4644 \\ \hline (0) \end{array}$$

Hence the ſecond Value of a is $= -86$, Or $86 = -a$ which ſeems impoſſible, *viz.* that an affirmative Quantity ſhould be equal to a negative Quantity ; yet even by this ſecond Value of a, and the ſame Co-efficient, the true (or firſt) Æquation may be formed.

Thus, Let	1	$a = -86$
1 ⊛ 2	2	$aa = +7396$, viz. $-86 \times -86 = +7396$
1 × 32	3	$32a = -2752$
2 + 3	4	$aa + 32a = 4644$ As at firſt.

Example 2.

Suppoſe	1	$aa - 7a = 948{,}75$ Then *per Theorem* 2.
1 C □	2	$aa - 7a + \frac{49}{4} = 948\ 75 + \frac{49}{4} = 961$
2 ω 2	3	$a - \frac{7}{2}$ (Or 3, 5) $= \sqrt{961} = 31$
3 + 3,5	4	$a = 31 + 3{,}5 = 34.5$

Again, for the ſecond Value of a,
Let $aa - 7a - 948{,}75 = 0$. And $a - 34{,}5 = 0$
Then, $a - 34{,}5 = 0)\ aa - 7a - 948{,}75 = 0\ (a + 27{,}5 = 0$
Conſequently this ſecond Value is $a = -27{,}5$
which will form the original Æquation, $aa - 7a = 948{,}75$ if it be ordered as the laſt was.

Example 3.

Suppoſe $36a - aa = 243$ Then *per Theorem* 3.
$a = 18 - \sqrt{324 - 243}$ *viz.* half Squared is 324 &c.
That is, $a = 18 - \sqrt{81}$ but $\sqrt{81} = 9$
Therefore $a = 18 - 9 = 9$. Now this third Form is called an ambiguous Æquation, becauſe it hath two affirmative Values of the unknown Quantity (a), both which may be found without ſuch Diviſion, as was uſed before,

For

For in this Cafe, $a = 18 \pm \sqrt{81}$, viz. $a = 18 + 9 = 27$. Or, $a = 18 - 9 = 9$, As before; And both thefe Values of a are equally true, as to forming the given Æquation;

Viz. $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, If $a = 27$, then will $aa = 729$ and $36a = 972$:
But $972 - 729 = 243$, confequently it may be, $a = 27$.

Now either of thefe Values of a may be found by Divifion, as thofe were in the other two Cafes, one of them being firft found by the *Theorem*.

Thus, Let $36a - aa - 243 = 0$ And $9 - a = 0$

$$\begin{array}{l} \text{Then } 9 - a = 0)\ 36a - aa - 243 = 0\ (a - 27 = 0 \\ \qquad\qquad\qquad \underline{9a - aa\qquad\quad} \\ \qquad\qquad\qquad 27a - 0 - 243 \\ \qquad\qquad\qquad \underline{27a \qquad - 243} \\ \qquad\qquad\qquad (0) \qquad\quad (0) \end{array}$$

Hence, if $a - 27 = 0$ Then $a = 27$ As before.

Notwithftanding all Quadratic Æquations of this third Form have two Affirmative Roots, (as in this) yet but one of thofe Roots will give a true Anfwer to the Queftion, and that is to be chofen according to the Nature and Limits of the Queftion, as fhall be fhewed further on.

SCHOLIUM.

From the Work of the three laft Examples, it may be obferved; that the Sum of both the Roots, will always be equal to the Co-efficient of their refpective Æquations, with a contrary Sign.

Thus. In *Example* 1. $aa + 32a = 4644$

$$\begin{array}{ll} \text{Here} & a = \quad 54 \\ \text{And} & \underline{a = -86} \end{array} \Big\}\ \text{Add}$$
$$2a = -32$$

In *Example* 2. $aa - 7a = 948{,}75$

$$\begin{array}{ll} \text{Here} & a = \quad 34{,}5 \\ \text{And} & \underline{a = -27{,}5} \end{array} \Big\}\ \text{Add}$$
$$2a = +7$$

In the laft *Example* $36a - aa = 243$
Which was changed into $aa - 36a = -243$

$$\begin{array}{ll} \text{Here} & a = 9 \\ \text{And} & \underline{a = 27} \end{array} \Big\}\ \text{Add}$$
$$2a = 36$$

Hence

Hence it is evident, that if either of the Roots be found, the other may be easily had without Division.

If the Contents of this Section be well understood, it will be easy to give a Numerical Solution to any Quadratic Æquation, that happens to arise in resolving of Questions, &c. And as for giving a Geometrical Construction of them, I think it not proper in this Place; because I here suppose the Learner wholly ignorant of the first Principles of Geometry, therefore I shall refer that Work to the next Part.

CHAP. IX.

Of Analysis, *or the Method of Resolving* Problems; *Exemplified by Variety of Numerical* Questions.

N. B. *Here I advise the young Learner to make use always of the same Letters, to represent the same* Data *in all Questions.*

Viz. $\left\{\begin{array}{l}\text{If } a \text{ represent any Number} \\ \text{And } e \text{ represent a less Number}\end{array}\right\}$ or other *Quantity.*

Then let $\left\{\begin{array}{ll} a+e=s & \text{Their Sum.} \\ a-e=d & \text{Their Difference.} \\ ae=p & \text{Their Product.} \\ \dfrac{a}{e}=q & \text{Their Quotient.} \\ aa+ee=z & \text{The Sum of their Squares.} \\ aa-ee=x & \text{The Difference of their Squares.}\end{array}\right.$

Any two of these six, (s, d, p, q, z, x) being given, thence to find the Rest; which admits of fifteen Variations, or Questions.

Question 1. *Suppose* s and d were given, and it were required by them to find $a . e . p . q . z .$ and x.

Let $\{$	1	$a+e=s$
	2	$a-e=d$ $\}$ and suppose $\left\{\begin{array}{l}s=240\\d=192\end{array}\right\}$ Then
$1+2$	3	$2a=s+d=432$
$3\div 2$	4	$a=\dfrac{s+d}{2}=216$ Here a is found.
$1-2$	5	$2e=s-d=48.$

$5\div 2$

5 ÷ 2	6	$e = \frac{s-d}{2} = 24$ Here e is found.
4 × 6	7	$ae = \frac{ss-dd}{4} = p = 5184$. Here p is found.
4 ÷ 6	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$ Here q is found.
4 ⊛ 2	9	$aa = \frac{ss+2sd+dd}{4} = 46656$.
6 ⊛ 2	10	$ee = \frac{ss-2sd+dd}{4} = 576$.
9 + 10	11	$aa+ee = \frac{ss+dd}{2} = z = 47232$. z found.
9 − 10	12	$aa-ee = sd = x = 46080$. x found.

Question 2. *Let s and p be given*; To find the *Rest*.

That is, {	1	$a+e = s. = 240$ } Quere $a.\ e.\ d.\ q.\ z.\ x.$
	2	$ae = p = 5184$
1 ⊛ 2	3	$aa+2ae+ee = ss = 57600$
2 × 4	4	$4ae = 4p = 20736$
3 − 4	5	$aa-2ae+ee = ss-4p = 36864$
5 ω 2	6	$a-e = \sqrt{ss-4p} = d = 192$
1 + 6	7	$2a = s+\sqrt{ss-4p}$
7 ÷ 2	8	$a = \frac{s+\sqrt{ss-4p}}{2}$ Hence $a = 216$.
1 − 6	9	$2e = s-\sqrt{ss-4p}$
9 ÷ 2	10	$e = \frac{s-\sqrt{ss-4p}}{2}$ Hence $e = 24$.
8 ÷ 10	11	$\frac{a}{e} = \frac{s+\sqrt{ss-4p}}{s-\sqrt{ss-4p}} = q = 9$.
8 ⊛ 2	12	$aa = \frac{ss+s\sqrt{ss-4p}}{2} : -p$
10 ⊛ 2	13	$ee = \frac{ss-s\sqrt{ss-4p}}{2} : -p$

12 + 13	14	$aa + ee = ss - 2p = z = 47232$
12 — 13	15	$aa - ee = s\sqrt{ss - 4p} = x = 46080$

Question 3. *Suppose* s and q are *given*; To find the *Rest.*

Viz.	1	$a + e = s. = 240$ Quere $a.e.d.p.z.x.$
	2	$\frac{a}{e} = q = 9$
$2 \times e$	3	$a = qe$
1 — 3	4	$e = s - qe$
$4 + qe$	5	$qe + e = s$
$5 \div q + 1$	6	$e = \frac{s}{q + 1}$ For $\overline{q + 1} \times e = qe + e$
1 — 6	7	$a = s - \frac{s}{q + 1} = \frac{qs}{q + 1}$
6×7	8	$ae = \frac{qss}{qq + 2q + 1} = p.$
7 — 6	9	$a - e = \frac{qs - s}{q + 1} = d$
7 Ⓖ 2	10	$aa = \frac{qqss}{qq + 2q + 1}$
6 Ⓖ 2	11	$ee = \frac{ss}{qq + 2q + 1}$
10 + 11	12	$aa + ee = \frac{qqss + ss}{qq + 2q + 1} = z$
10 — 11	13	$aa - ee = \frac{qqss - ss}{qq + 2q + 1} = x$

Question 4. Let s and z be *given*; To find the *Rest.*

Viz.	1	$a + e = s. = 240$ Quere $a.e.d.p.q.x.$
	2	$aa + ee = z. = 47232$
1 Ⓖ 2	3	$aa + 2ae + ee = ss$
3 — 2	4	$2ae = ss - z$
2 — 4	5	$aa - 2ae + ee = 2z - ss$
5 ш 2	6	$a - e = \sqrt{2z - ss} = d$

1 + 6

$1+6$	7	$2a = s + \sqrt{2z - ss}$	The Rest are found just as in the 2d Question; the 8 and 10 Steps here, being the very same with the 8 and 10 Steps there.
$7 \div 2$	8	$a = \dfrac{s + \sqrt{2z - ss}}{2}$	
$1 - 6$	9	$2e = s - \sqrt{2z - ss}$	
$9 \div 2$	10	$e = \dfrac{s - \sqrt{2z - ss}}{2}$	

Question 5. When s and x are *given*; To find the *Rest*.

Viz.	1	$a + e = s = 240$
	2	$aa - ee = x = 46080$ Quere $a . e . d . p . q . z.$
$2 \div 1$	3	$a - e = \dfrac{x}{s} = d$ viz. $a + e\,)\ aa - ee\ (a - e$
$1 + 3$	4	$2a = s + \dfrac{x}{s} = \dfrac{ss + x}{s}$
$4 \div 2$	5	$a = \dfrac{ss + x}{2s}$
$1 - 3$	6	$2e = s - \dfrac{x}{s} = \dfrac{ss - x}{s}$
$6 \div 2$	7	$e = \dfrac{ss - x}{2s}$
5×7	8	$ae = \dfrac{ssss - xx}{4ss} = p$
$5 \div 7$	9	$\dfrac{a}{e} = \dfrac{ss + x}{ss - x} = q$
5 ⊛ 2	10	$aa = \dfrac{s^4 - 2ssx + xx}{4ss}$
7 ⊛ 2	11	$ee = \dfrac{s^4 - 2ssx + xx}{4ss}$
$10 + 11$	12	$aa + ee = \dfrac{s^4 + xx}{2ss} = z$

Question 6. *Suppose* d and p are *given*; To find the *Rest*.

Viz.	1	$a - e = d = 192$
	2	$ae = p = 5184$ Quere $a . e . s . q . z . x.$

1 Ⓖ 2	3	$aa - 2ae + ee = dd$
$2 \times \bar{4}$	4	$4ae = 4p$
$3 + 4$	5	$aa + 2ae + ee = dd + 4p$
5 ω 2	6	$a + e = \sqrt{dd + 4p} = s$
$6 + 1$	7	$2a = d + \sqrt{dd + 4p}$
$7 \div \bar{2}$	8	$a = \dfrac{d + \sqrt{dd + 4p}}{2}$
$6 - 1$	9	$2e = \sqrt{dd + 4p} : - d$
$9 \div \bar{2}$	10	$e = \dfrac{\sqrt{dd + 4p} : - d}{2}$
$8 \div 10$	11	$\dfrac{a}{e} = \dfrac{d + \sqrt{dd + 4p}}{\sqrt{dd + 4p} : - d} = q$
8 Ⓖ 2	12	$aa = \dfrac{dd + 2p + d\sqrt{dd + 4p}}{2}$
10 Ⓖ 2	13	$ee = \dfrac{dd + 2p - d\sqrt{dd + 4p}}{2}$
$12 + 13$	14	$aa + ee = dd + 2p = z$
$12 - 13$	15	$aa - ee = d\sqrt{dd + 4p} = x$

Question 7. *Let d* and *q* be *given*; To find the *Rest*.

Viz.	1	$a - e = d$ 192
	2	$\dfrac{a}{e} = q =$ 9 — Quere $a . e . s . p . z . x$.
$2 \times e$	3	$a = qe$
$1 + e$	4	$a = d + e$
3, 4	5	$qe = d + e$
$5 - e$	6	$qe - e = d$
$6 \div q - 1$	7	$e = \dfrac{d}{q - 1}$ For $q - 1 \times e = qe - e$
$1 + 7$	8	$a = d + \dfrac{d}{q - 1} = \dfrac{qd}{q - 1}$
$7 + 8$	9	$a + e = \dfrac{qd + d}{q - 1} = s$

7×8	10	$ae = \frac{qdd}{qq - 2q + 1} = p$
$8 \circledast 2$	11	$aa = \frac{qqdd}{qq - 2q + 1}$
$7 \circledast 2$	12	$ee = \frac{dd}{qq - 2q + 1}$
$11 + 12$	13	$aa + ee = \frac{qqdd + dd}{qq - 2q + 1} = z$
$11 - 12$	14	$aa - ee = \frac{qqdd - dd}{qq - 2q + 1} = x$

Question 8. *Suppose d* and *z given*; To find the *Rest.*

Viz.	1	$a - e = d = 192$ Quere $a.e.s.p.q.x.$
	2	$aa + ee = z = 47232$
$1 \circledast 2$	3	$aa - 2ae + ee = dd$
$2 - 3$	4	$2ae = z - dd$
$2 + 4$	5	$aa + 2ae + ae = 2z - dd$
$5\ \omega\ 2$	6	$a + e = \sqrt{2z - dd} = s$
$1 + 6$	7	$2a = d + \sqrt{2z - dd}$
$7 \div 2$	8	$a = \frac{d + \sqrt{2z - dd}}{2}$
$6 - 1$	9	$2e = \sqrt{2z - dd} : - d$
$9 \div 2$	10	$e = \frac{\sqrt{2z - dd} : - d}{2}$
8×10	11	$ae = \frac{z - dd}{2} = p$
$8 \circledast 2$	12	$aa = \frac{z + d\sqrt{2z - dd}}{2}$
$10 \circledast 2$	13	$ee = \frac{z - d\sqrt{2z - dd}}{2}$
$12 - 13$	14	$aa - ee = d\sqrt{2z - dd} = x$
$8 \div 10$	15	$\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} : - d} = q$

Question

Question 9. *Let* d and x be *given*; To find the *Rest*:

Viz.	1	$a - e = d = 240$ } Quere $a.e.s.p.q.z.$
	2	$aa - ee = x = 46080$ }
$2 \div 1$	3	$a + e = \frac{x}{d} = s.$ viz. $a - e)\ aa - ee\ (a + e$
$1 + 3$	4	$2a = \frac{dd + x}{d}$
$4 \div 2$	5	$a = \frac{dd + x}{2d}$
$3 - 5$	6	$e = \frac{x - dd}{2d}$
5×6	7	$ae = \frac{xx - d^4}{4dd} = p$
$5 \div 6$	8	$\frac{a}{e} = \frac{dd + x}{x - dd} = q$
5 ⊚ 2	9	$aa = \frac{d^4 + 2ddx + xx}{4dd}$
6 ⊚ 2	10	$ee = \frac{xx - 2ddx + d^4}{4dd}$
$9 + 10$	11	$aa + ee = \frac{d^4 + xx}{2dd} = z$

Question 10. *Let* p and q be *given*; To find the *Rest*.

Viz.	1	$ae = p = 5184$ } Quere $a.e.s.d.z.x.$
	2	$\frac{a}{e} = q = 9$ }
1×2	3	$aa = qp$ For $\frac{ae}{1} \times \frac{a}{e} = \frac{aae}{e} = aa$
3 ω 2	4	$a = \sqrt{qp}$
$1 \div 2$	5	$ee = \frac{p}{q}$ For $\frac{a}{e})\ \frac{ae}{1}\ (\frac{aee}{a} = ee$
5 ω 2	6	$e = \sqrt{\frac{p}{q}}$
$4 + 6$	7	$a + e = \sqrt{qp} + \sqrt{\frac{p}{q}} = s$

4 — 6

4 − 6	8	$a - e = \sqrt{qp} - \sqrt{\frac{p}{q}} = d$
3 + 5	9	$aa + ee = qp + \frac{p}{q} = z$
3 − 5	10	$aa - ee = qp - \frac{p}{q} = x$

Question 11. *Let* p and z be *given*; To find the *Rest*.

Viz.	1	$ae = p = 5184$
	2	$aa + ee = z = 47232$ — Quere *a*. *e*. &c.
1 × 2	3	$2ae = 2p$
2 + 3	4	$aa + 2ae + ee = z + 2p$
4 ɯ 2	5	$a + e = \sqrt{z + 2p} = s$
2 − 3	6	$aa - 2ae + ee = z - 2p$
6 ɯ 2	7	$a - e = \sqrt{z - 2p} = d$
5 + 7	8	$2a = \sqrt{z + 2p} + \sqrt{z - 2p}$
8 ÷ 2	9	$a = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{2}$
5 − 7	10	$2e = \sqrt{z + 2p} - \sqrt{z - 2p}$
10 ÷ 2	11	$e = \frac{\sqrt{z + 2p} - \sqrt{z - 2p}}{2}$
9 ÷ 11	12	$\frac{a}{e} = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{\sqrt{z + 2p} - \sqrt{z - 2p}}$
9 Ⓖ 2	13	$aa = \frac{z + \sqrt{zz - 4pp}}{2}$
11 Ⓖ 2	14	$ee = \frac{z - \sqrt{zz - 4pp}}{2}$
$aa - ee$	15	$aa - ee = \sqrt{zz - 4pp} = x$

Question 12. *Let* p and x be *given*; To find the *Rest*.

Viz.	1	$ae = p = 5184$
	2	$aa - ee = x = 46080$ — Quere *a*. *e*. &c.
1 Ⓖ 2	3	$aaee = pp$

2 ⊚ 2	4	$aaaa - 2aaee + eeee = xx.$
3 × 4	5	$4aaee = 4pp$
4 + 5	6	$aaaa + 2aaee + eeee = xx + 4pp$
6 ɯ 2	7	$aa + ee = \sqrt{xx + 4pp} = z$
2 + 7	8	$2aa = x + \sqrt{xx + 4pp}$
8 ÷ 2	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
9 ɯ 2	10	$a = \sqrt{:\frac{x + \sqrt{xx + 4pp}}{2}}$
7 − 2	11	$2ee = \sqrt{xx + 4pp} : - x$
11 ÷ 2	12	$ee = \frac{\sqrt{xx + 4pp} : - x}{2}$
12 ɯ 2	13	$e = \sqrt{:\frac{\sqrt{xx + 4pp} : - x}{2}}$
10 + 13	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp}. - x}{2}} = s$
10 − 13	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = d$
9 + 12	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. *Having* q and z *given*; To find the *Rest*.

Viz.	1	$\frac{a}{e} = q = 9$ Quere a. e. &c.
	2	$aa + ee = z = 47232$
1 × e	3	$a = qe$
3 ⊚ 2	4	$aa = qqee$
2 − 4	5	$ee = z - qqee$
4 + qqee	6	$qqee + ee = z$
6 ÷ qq + 1	7	$ee = \frac{z}{qq + 1}$ For $qq + 1 \times ee = qqee + ee$

2 − 7

$2 - 7$	8	$aa = z - \frac{z}{qq+1} = \frac{qqz}{qq+1}$
$8 \omega 2$	9	$a = \sqrt{\frac{qqz}{qq+1}}$
$7 \omega 2$	10	$e = \sqrt{\frac{z}{qq+1}}$
$9 + 10$	11	$a+e = \sqrt{\frac{qqz}{qq+1}} : + \sqrt{\frac{z}{qq+1}} = s$
$9 - 10$	12	$a-e = \sqrt{\frac{qqz}{qq+1}} : - \sqrt{\frac{z}{qq+1}} = d$
9×10	13	$ae = \sqrt{\frac{qqzz}{q^4+2qq+1}} = p$
$8 - 7$	14	$aa - ee = \frac{qqz - z}{qq+1} = x$

Question 14. When q and x are *given*; To find the *Rest*.

Viz.	1	$\frac{a}{e} = q = 9$ } Quere a. e. &c.
	2	$aa - ee = x = 46080$
$1 \times e$	3	$a = qe$
$3 \odot 2$	4	$aa = qqee$
$2 + ee$	5	$aa = x + ee$
$4, 5$	6	$qqee = x + ee$
$6 - ee$	7	$qqee - ee = x$
$7 \div qq - 1$	8	$ee = \frac{x}{qq-1}$
$2 + 8$	9	$aa = x + \frac{x}{qq-1} = \frac{qqx}{qq-1}$
$9 \omega 2$	10	$a = \sqrt{\frac{qqx}{qq-1}}$
$8 \omega 2$	11	$e = \sqrt{\frac{x}{qq-1}}$
$10 + 11$	12	$a+e = \sqrt{\frac{qqx}{qq-1}} : + \sqrt{\frac{x}{qq-1}} = s$

$10-11$	13	$a-e=\sqrt{\dfrac{qqx}{qq-1}}-\sqrt{\dfrac{x}{qq-1}}=d$
10×11	14	$ae=\sqrt{\dfrac{qqxx}{qqqq-2qq+1}}=p$
$8+9$	15	$aa+ee=\dfrac{qqx+x}{qq-1}=z$

Question 15. *When* z and x are *given*; To find the *Rest.*

Viz.	1	$aa+ee=z=47232$ Quere a. e. &c.
	2	$aa-ee=x=46080$
$1+2$	3	$2aa=z+x$
$3\div2$	4	$aa=\dfrac{z+x}{2}$
$1-2$	5	$2ee=z-x$
$5\div2$	6	$ee=\dfrac{z-x}{2}$
4 ѡ 2	7	$a=\sqrt{\dfrac{z+x}{2}}$
6 ѡ 2	8	$e=\sqrt{\dfrac{z-x}{2}}$
$7+8$	9	$a+e=\sqrt{\dfrac{z+x}{2}}+\sqrt{\dfrac{z-x}{2}}=s$
$7-8$	10	$a-e=\sqrt{\dfrac{z+x}{2}}-\sqrt{\dfrac{z-x}{2}}=d$
7×8	11	$ae=\sqrt{\dfrac{zz-xx}{4}}=p$
$7\div8$	12	$\dfrac{a}{e}=\dfrac{\sqrt{z+x}}{\sqrt{z-x}}=q$

These fifteen Questions are proposed in Doctor *Pell's Algebra*; but he pursues only the first Question throughout, and breaks off in the other Fourteen, after the Values of what I call a and e are found. But I have proceeded in every one of them, to find the Values of all the unknown Quantities, because they afford such Variety, as being well observed by a Learner, will be found very useful in the Solution of most Questions.

Note,

Note, I have chose to use the same Numbers for the respective Value of each Quantity throughout all the Questions, because they will be more satisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleasure, provided that the Number represented by *a*, be greater than that by *e*, &c. I have omitted the Numerical Calculations purely for the Learner to practise on.

Question 16. *There are two Numbers, the Sum of their Squares is* 2368; *And the Greater of them is in Proportion to the Less, As* 6 *To* 1. *What are those Numbers?*

Let a = *the Greater Number,* e = *the Lesser, and* $z = 2368$.

Then	1	$aa + ee = z$ } *By the* Question.
And	2	$a : e :: 6 : 1$
2 ∵	3	$1a = 6e$
3 ⊗ 2	4	$aa = 36ee$
1 — 4	5	$ee = z - 36ee$
5 + 36ee	6	$37ee = z$
6 ÷ 37	7	$ee = \frac{z}{37} = 64$
7 w 2	8	$e = \sqrt{\frac{z}{37}} = 8$
8 × 6	9	$6e = 6\sqrt{\frac{z}{37}} = 48$
3 · 9	10	$a = 48$

Proof:
If $a = 48$
And $e = 8$
$aa = 2304$
$ee = 64$
$aa + ee = 2368$
And $48 : 8 :: 6 : 1$

Question 17. *There are three Numbers in Continued Proportion, the Sum of the Extremes is* 156, *and the Mean is* 72; *What are the two Extremes?*

That is, Suppose $a . m . e$ in ∺ and $m = 72$.

Then	1	$a + e = s = 156$ } *By the* Question.
	2	$a : m :: m : e$ } Quere $a . e$.
2 ∵	3	$ae = mm$
1 ⊗ 2	4	$aa + 2ae + ee = ss$
3 × 4	5	$4ae = 4mm$
4 — 5	6	$aa - 2ae + ee = ss - 4mm$
6 w 2	7	$a - e = \sqrt{ss - 4mm}$
1 + 7	8	$2a = s + \sqrt{ss - 4mm}$

8 ÷ 2

$8 \div 2$	9	$a = \dfrac{s + \sqrt{ss - 4mm}}{2} = 108$	Or $a = 48$
$1 - 9$	10	$e = \dfrac{s - \sqrt{ss - 4mm}}{2} = 48$	$e = 108$

Question 18. *There are three Numbers in ÷ their Sum is* 74, *and the Sum of their Squares is* 1924; *What are those Numbers?*

That is, a, e, y are in ÷.

Then	1	$a + e + y = s = 74$	Quere a, e, y.
	2	$aa + ee + yy = z = 1925$	
	3	$a : e :: e : y$	
3 ∵	4	$ay = ee$	
$1 - e$	5	$a + y = s - e$	
$2 - ee$	6	$aa + yy = z - ee$	
4×2	7	$2ay = 2ee$	
$6 + 7$	8	$aa + 2ay + yy = z + ee$	
5 ⊙ 2	9	$aa + 2ay + yy = ss - 2se + ee$	
8 and 9	10	$z + ee = ss - 2se + ee$	
$10 \pm$	11	$2se = ss - z$	
$11 \div 2s$	12	$e = \dfrac{ss - z}{2s} = 24$	
5	13	$a + y = s - e = 50$	
13 ⊙ 2	14	$aa + 2ay + yy = 2500$	
4×4	15	$4ay = 4ee = 2304$	
$14 - 15$	16	$aa - 2ay + yy = 196$	
16 ω 2	17	$a - y = \sqrt{196} = 14$	
$13 + 17$	18	$2a = 50 + 14 = 64$	
$18 \div 2$	19	$a = 32$	Or $a = 18$
$13 - 19$	20	$y = 50 - 32 = 18$	$y = 32$

Note, In all Questions about Continual Proportionals, (either Arithmetical or Geometrical) where three Terms are sought, the Mean is easiest found first (as above;) and if all the Terms be affirmative, then 'tis equal whether the first, or last Term be the greatest.

Question 19. *There are three Numbers in ÷ their Sum is* 76; *and if the Sum of the Extremes be multiplied into the Mean, that Product will be* 1248; *What are those Numbers?*

Viz.

Viz.	1	$a : e :: e : y$
	2	$a + e + y = s = 76$ By the Question.
	3	$ae + ye = p = 1248$
1 ∵	4	$ay = ee$
$2 \times e$	5	$ae + ee + ye = se$
$5 - 3$	6	$ee = se - p$
$6 - se$	7	$ee - se = -p$
7 C □	8	$ee - se + \frac{1}{4}ss = \frac{1}{4}ss - p$
8 ω 2	9	$e - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$
$9 + \frac{1}{2}s.$	10	$e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = \begin{cases} 52. \\ 24. \end{cases}$ *Per Theor.* 3. *Chap.* 8.
$2 - 10$	11	$a + y = 52$
4×4	12	$4ay = 4ee = 2304$
11 ⊕ 2	13	$aa + 2ay + yy = 2704$
$13 - 12$	14	$aa - 2ay + yy = 400$
14 ω 2	15	$a - y = \sqrt{400} = 20$
$11 + 15$	16	$2a = 52 + 20 = 72$
$16 \div 2$	17	$a = 36$ Or $a = 16$
$11 - 17$	18	$y = 52 - 36 = 16$ and $y = 36$

N. B. If you take $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = 52$ (at the 10th Step) Then it will be $76 - 52 = 24 = a + y$, which is impossible, *viz.* that the Mean shou'd be greater than the Sum of the two Extremes.

Therefore it must be $e = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - p} = 24$. (*See Pag.* 201.

Question 20. *There are three Numbers in Arithmetical Progression, the First being added to twice the Second, and three times the Third, their Sum will be* 62; *and the Sum of all their Squares is* 275 : *What are those Numbers?*

Suppose	1	$a.\ e,\ y$ in *Arithmetical Progression.*
And	2	$a + 2e + 3y = 62$ by the Question.
	3	$aa + ee + yy = 275$
Then	4	$a + y = 2e$ *per Sect.* 1. *Chap.* 6.
$2 - 4$	5	$2e + 2y = 62 - 2e$
$5 \div 2$	6	$e + y = 31 - e$
$6 - e$	7	$y = 31 - 2e$
$4 - 7$	8	$a = 4e - 31$
8 ⊕ 2	9	$aa = 16ee - 248e + 961$
7 ⊕ 2	10	$yy = 961 - 124e + 4ee$
$9 + 10$	11	$aa + yy = 20ee - 372e + 1922$

3 — 11	12	$ee = 372e - 20ee - 1647$
12 + 20ee	13	$21ee = 372e - 1647$
13 — 37ee	14	$21ee - 372e = -1647$
14 ÷ $\overline{21}$	15	$ee - \frac{124}{7}e = -\frac{549}{7}$
15 C □	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{49}$
16 ω 2	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$ The Mean.
17 + $\frac{62}{7}$	18	$e = \frac{62}{7} \pm \frac{1}{7} = 9$ Or $8\frac{5}{7}$
18 × 4	19	$4e = 36$ Or $8\frac{6}{7}$
8, 19	20	$a = 36 - 31 = 5$ Or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
18 × $\overline{2}$	21	$2e = 18$ Or $17\frac{3}{7}$
7, 21	22	$y = 31 - 18 = 13$ Or $31 - 17\frac{3}{7} = 13\frac{4}{7}$

Question 21. *There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is 576: The Square of the Mean being added to the Product of the two Extremes, makes 612: And the Square of the last Term being added to the Product of the first into the second, is 792: What are those Numbers?*

Suppose	1	a, e, y In *Arith. Progres.* *As* before.
Then	2	$aa + ye = 576$ By the Question.
	3	$ee + ya = 612$
	4	$yy + ae = 792$
1 ∵	5	$a + y = 2e$ *Per Sect. 1. Chap. 6.*
5 × e	6	$ae + ye = 2ee$
2 + 4	7	$aa + ye + yy + ae = 1368$
7 — 6	8	$aa + yy = 1368 - 2ee$
3 — ee	9	$ya = 612 - ee$
9 × 2	10	$2ya = 1224 - 2ee$
8 + 10	11	$aa + 2ya + yy = 2592 - 4ee$
5 ⊛ 2	12	$aa + 2ya + yy = 4ee$
11, 12	13	$4ee = 2592 - 4ee$
13 + 4ee	14	$8ee = 2592$
14 ÷ 8	15	$ee = 324$
15 ω 2	16	$e = \sqrt{324} = 18$ The Mean.
8,	17	$aa + yy = 1368 - 2ee = 720$
10,	18	$2ya = 1224 - 2ee = 576$
17 — 18	19	$aa - 2ya + yy = 720 - 576 = 144$

1 ꟷ 2	20	$a - y = \sqrt{144} = 12$
$5 + 20$	21	$2a = 2e + 12 = 48$
$21 \div 2$	22	$a = 24$
$5 - 22$	23	$y = 2e - 24 = 12$

Or $\begin{cases} a = 12 \\ y = 24 \end{cases}$

Question 22. *'Tis required to find two such Numbers, that the Sum of their Squares may be* $8226\frac{1}{2}$; *and their Product being added to the Square of the Lesser, may be* $6921\frac{1}{2}$.

Viz.	1	$aa + ee = 8226\frac{1}{2}$ } Quere *a* and *e*
	2	$ae + ee = 6921\frac{1}{2}$
$1 - 2$	3	$aa - ae = 1305$
$3 \pm$	4	$ae = aa - 1305$
$4 \div a$	5	$e = \dfrac{aa - 1305}{a}$
5 ⚭ 2	6	$ee = \dfrac{a^4 - 2610aa + 1703025}{aa}$
$1 - aa$	7	$ee = 8226{,}5 - aa$
6, 7	8	$\dfrac{a^4 - 2610aa + 1703025}{aa} = 8226{,}5 - aa$
$8 \times aa$	9	$a^4 - 2610aa + 1703025 = 8226{,}5aa - a^4$
$9 + a^4$	10	$2a^4 - 2610aa + 1703025 = 8226{,}5\,aa$
$10 \pm$	11	$2a^4 - 10836{,}5aa = -1703025$
$11 \div 2$	12	$a^4 - 5418.25aa = -851512{,}5$
12 C□	13	$a^4 - 5418{,}25aa + 733958.26562 = 6487845{,}765625$
13 ꟷ 2	14	$aa - 2709{,}125 = \sqrt{6487845{,}765625} = 2547{,}125$
$14 + \overline{27}$ &c.	15	$aa = 2709{,}125 \pm 2547.125$
Suppose	16	$aa = 2709{,}125 + 2547{,}125 = 5256{,}25$
Then	17	$a = \sqrt{5256{,}25} = 72.5$
And 5,	18	$e = \dfrac{aa + 1305}{a} = \dfrac{5256.25 - 1305}{72.5} = 54{,}5$
Or let	19	$aa = 2709{,}125 - 2547.125 = 162$
10 ꟷ 2	20	$a = \sqrt{162} = 12.72$ &c.
Then	21	$e = \dfrac{162 - 1305}{12{,}72}$ Which is *impossible.*
Therefore		$a = 72{,}5$ } As at the 17th and 18th Steps.
And		$e = 54.5$

This *Question* may be perform'd with less Trouble, by *Substituting Letters* for the known *Numbers.*

Viz. $\begin{cases} aa + ee = z \\ ae + ee = p \end{cases}$ Then let $z - p = d = aa - ae$, &c.

Question

Question 23. *It is required to find three such Numbers, that the Sum of the First and Second, being multiplied with the Third, may be* 37824; *and the Sum of the Second and Third, multiplied with the First, may be* 59944, *also, that the Sum of the First and Third, being multiplied with the Second, may be* 52456.

Let a, e, y represent the *three Numbers*.

Then	1	$ay + ey = 37824 = b$
	2	$ea + ya = 59944 = c$ — Quere a, e, y.
	3	$ae + ye = 52456 = d$
1 + 2 + 3	4	$2ae + 2ay + 2ye = b + c + d$
Let	5	$z = b + c + d$
4 ÷ 2	6	$ae + ay + ye = \frac{1}{2}z = \frac{b+c+d}{2}$
6 − 3	7	$ay = \frac{1}{2}z - d$
7 ÷ a	8	$y = \frac{z - 2d}{2a}$
6 − 2	9	$ye = \frac{1}{2}z - c$
6 − 1	10	$ae = \frac{1}{2}z - b = \frac{z - 2b}{2}$
10 ÷ a	11	$e = \frac{z - 2b}{2a}$
8 × 11	12	$ye = \frac{z - 2d}{2a} \times \frac{z - 2b}{2a} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$
9, 12	13	$\frac{z - 2c}{2} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$
13 × $4aa$	14	$2zaa - 4caa = zz - 2dz - 2bz + 4bd$
14 ÷	15	$aa = \frac{zz - 2dz - 2bz + 4bd}{2z - 4c} = 55696$
15 ɯ 2	16	$a = \sqrt{55696} = 236$
11	17	$e = \frac{z - 2b}{2a} = 158$
8,	18	$y = \frac{z - 2d}{2a} = 96$

Question 24. *'Tis required to find two such Numbers, that their Sum, being subtracted from the Sum of their Squares, may leave* 14. *And if their Product be added to their Sum, it may make* 14.

Let

Let a and e be put for the *Numbers*, and let $y = a + e$

Then	1	$aa + ee - y = 14$ } By the *Question.*
	2	$ae + y = 14$
$1 + y$	3	$aa + ee = 14 + y$
$2 - y$	4	$ae = 14 - y$
4×2	5	$2ae = 28 - 2y$
$3 + 5$	6	$aa + 2ae + ee = 42 - y$
6 ω 2	7	$a + e = \sqrt{42 - y}$
But	8	$a + e = y$ By *Substitution* above.
7, 8,	9	$y = \sqrt{42 - y}$
9 ⊙ 2	10	$yy = 42 - y$
$10 + y$	11	$yy + y = 42$
11 C □	12	$yy + y + \frac{1}{4} = 42 + \frac{1}{4} = 42{,}25$
12 ω 2	13	$yy + \frac{1}{2} = \sqrt{42{,}25} = 6{,}5$
$13 - \frac{1}{2}$	14	$y = 6{,}5 - \frac{1}{2} = 6$
Consequent	15	$a + e = 6$ By *Restitution* from above.
3, 14	16	$aa + ee = 14 + 6 = 20$
5, 15	17	$2ae = 28 - 12 = 16$
$16 - 17$	18	$aa - 2ae + ee = 4$
18 ω 2	19	$a - e = \sqrt{4} = 2$
$15 + 19$	20	$2a = 8$
$23 \div 2$	21	$a = 4$
$15 - 21$	22	$e = 6 - 4 = 2$

Proof: If $a = 4$ And $e = 2$
Then $aa + ee - a - e = 14$
And $ae + a + e = 14$
According to the *Question.*

Question 25. *Three Men discoursing of their Money, saith the First, if* 100 l. *were added to my Money, it would be as much as both your Money put together; said the second Man, if* 100 l. *were added to my Money, I should have twice as much as both you have; saith the third Man, if* 100 l. *were added to my Money, I should have then three times as much Money as both you have: How much Money had each Man?*

Let a represent the first Man's Money, e the Second, and y the Third?

Then	1	$a + 100 = e + y$ } By the *Question.*
	2	$e + 100 = 2a + 2y$
	3	$y + 100 = 3a + 3e$
$1 - a$	4	$e + y - a = 100 = s$ } Quere a, e, y.
$2 - e$	5	$2a + 2y - e = 100 = s$
$3 - y$	6	$3a + 3e - y = 100 = s$
4, 6	7	$e + y - a = 3a + 3e - y$
$7 \pm$	8	$2y = 4a + 2e$
$5 - 8$	9	$2a - e = s - 4a - 2e$
$9 + 4a$	10	$6a + e = s = 100$

4 + 6	11	$2a + 4e = 2s = 200$
10 × 4	12	$24a + 4e = 4s = 400$
12 − 11	13	$22a = 2s = 200$
13 ÷ 22	14	$a = \frac{s}{11} = \frac{100}{11} = 9\frac{1}{11}\,l.$
10 − 6a	15	$e = s - 6a = 100 - \frac{600}{11} = \frac{500}{11} = 45\frac{5}{11}\,l.$
8 ÷ 2	16	$y = 2a + e = \frac{200}{11} + \frac{500}{11} = \frac{700}{11} = 63\frac{7}{11}\,l.$

Answer. The { *First* / *Second* / *Third* } *Man* had { $9\,l.\ 1\,s.\ 9\frac{9}{11}\,d.$ / $45\,l.\ 9\,s.\ 1\frac{1}{11}\,d.$ / $63\,l.\ 12\,s.\ 8\frac{8}{11}\,d.$ }

Queſtion 26. *Three Men have each ſuch a Sum of Money, that if the firſt and ſecond Mens Money be added to half of what the third Man hath; that Sum will be 92* l. *And if the ſecond and third Mens Money be added to one third Part of the firſt Man's Money, that Sum will be 92* l. *Laſtly, if one fourth Part of the ſecond Man's Money be added to the firſt and third Mens Money, that Sum will alſo be 92* l. *How much was each Man's Money?*

Put a for the firſt Man's Money, e for the Second, and y for the Third.

Then	1	$a + e + \frac{1}{2}y = s$ By the *Queſtion*. And $s = 92$
	2	$\frac{1}{3}a + e + y = s$
	3	$\frac{1}{4}e + a + y = s$
1, 2	4	$a + e + \frac{1}{2}y = \frac{1}{3}a + e + y$
4 − e	5	$a + \frac{1}{2}y = \frac{1}{3}a + y$
5 × 2 × 3	6	$6a + 3y = 2a + 6y$
6 ±	7	$4a = 3y$
2 × 3	8	$a + 3e + 3y = 3s.$
8 − 7	9	$a + 3e = 3s - 4a$
9 − a	10	$3e = 3s - 5a$
10 ÷ 3	11	$e = \frac{3s - 5a}{3}$
3 × 4	12	$e + 4a + 4y = 4s = 368$
12 − 2	13	$3\frac{2}{3}a + 3y = 3s = 276$
13, 7	14	$3\frac{2}{3}a + 4a = 3s = 276$
14 × 3	15	$11a + 12a = 9s = 828$
15 ÷ 23	16	$a = \frac{9s}{23} = \frac{828}{23} = 36\,l.$ The 1ſt Man's Money.
11	17	$e = \frac{3s - 5a}{3} = \frac{96}{3} = 32\,l.$ The 2d Man's Money.
7 ÷ 3	18	$y = \frac{4a}{3} = \frac{144}{3} = 48\,l.$ The 3d Man's Money.

Question 27. *Four Men walking abroad found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found; that if the First took 25 Shillings from the Second, it would make his Number equal with what the Second had then left. If the Second took 30 Shillings from the Third, his Money would then be triple to what the Third had left. And if the Third took 40 Shillings from the Fourth, his Money would then be double to what the Fourth had left. Lastly, the Fourth taking 50 Shillings from the First, he would then have three times as much as the First had left, and 5 Shillings more.*

'Tis required to tell how many Shillings each Man had? put a for the first Sum, e the Second, y the Third, and u the Fourth.

Then	1	$a+25=e-25$	By the *Question.*
	2	$e+30=3y-90$	
	3	$y+40=2u-80$	
	4	$u+50=3a-145$	
$1+25$	5	$a+50=e$	
$2-30$	6	$3y-120=e$	
$5 . 6$	7	$a+50=3y-120$	
$7+120$	8	$a+170=3y$	
$8 \div 3$	9	$y=\frac{a+170}{3}$	
$3-40$	10	$y=2u-120$	
$9, 10$	11	$2u-120=\frac{a+170}{3}$	
$11+120$	12	$2u=\frac{a+170}{3}+120=\frac{a+530}{3}$	
$12 \div 2$	13	$u=\frac{a+530}{6}$	
$4-50$	14	$u=3a-195$	
$13 . 14$	15	$3a-195=\frac{a+530}{6}$	
15×6	16	$18a-1170=a+530$	
$16 \pm$	17	$17a=1700$	
$17 \div 17$	18	$a=100$	The 1st
by the 5	19	$e=150$	Second
by the 9	20	$y=90$	Third — *Man's N. of Shillings.*
by the 14	21	$u=105$	Fourth

Question

Question 28. *Four Men have each a Sum of Money which being put all together makes* 250 *Pounds. And if to the first Man's Money be added* 8 *Pounds it will be just as much as the second Man's Money decreased by* 8 *Pounds, and as much as* 8 *times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; How much had each Man?*

Let a, e, y, u represent the *Four Mens Money*.

Then	1	$a+e+y+u=s$
	2	$a+b=e-b$
	3	$yb=\frac{u}{b}=a+b$

by the Question. Let $s=250$ and $b=8$. Or any other Number at Pleasure.

$2+b$	4	$a+2b=e$
$3\div b$	5	$y=\frac{a+b}{b}$ Because $yb=a+b$
$3\times b$	6	$u=ba+bb$. For $\frac{u}{b}=a+b$
$4+5+6$	7	$e+y+u=a+2b+\frac{a+b}{b}+ba+bb$
$1-a$	8	$e+y+u=s-a$
7, 8	9	$a+2b+\frac{a+b}{b}+ba+bb=s-a$
$9\times b$	10	$ba+2bb+a+b+bba+bbb=bs-ba$
$10\pm$	11	$2ba+bba+a=bs-bbb-2bb-b$
$11\div$	12	$a=\frac{bs-bbb-2bb-b}{bb+2b+1}=16{,}691358$, &c.
by the 4.	13	$e=a+2b=32{,}691358$, &c.
by the 5.	14	$y=\frac{a+b}{b}=3{,}086419$, &c.
by the 6.	15	$u=ba+bb=197{,}530864$, &c.

That is		*l.*	*s.*	*d.*
	$a=$	16	13	9,92592
	$e=$	32	13	9,92592
	$y=$	3	1	8,74056
	$u=$	197	10	7,40736

Consequently $a+e+y+u=249$. 19 . 11,99976 which should be just 250*l.* the Sum proposed in the Question. Now what it wants of that Sum proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, tho' not perhaps exactly so. *Vide Sect.* 5. *Chap.* 5. *Part* I.

Question 29. *Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as there were Partners; with that Stock they traded, and gain'd as many Pounds per* 100 l. *as there were Partners. Now if* 10 l. 10 s. *be added to, and subtracted from their Gain, the Product of that Sum, and Difference will be* 6491 l. 6 s. 3 d.

Quere, How many *Merchants* there were, *&c.*

Let	1	$a =$ The Number of Merchants.
1×65	2	$65a =$ Every one's Sum he put into *Stock*.
$2 \times a$	3	$65aa =$ The whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$ by the *Question*.
Viz.	5	$\frac{65aaa}{100} =$ The whole *Gain*.
$5 + 10,5$	6	$\frac{65aaa}{100} + 10,5$
$5 - 10,5$	7	$\frac{65aaa}{100} - 10,5$
6×7	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$ by the *Question*.
8×10000	9	$4225a^6 - 1102500 = 64913125$
$9 +$	10	$4225a^6 = 66015625$
$10 \div 4225$	11	$a^6 = \frac{66015625}{4225} = 15625$
11 ω 6	12	$a = \sqrt[6]{15625} = 5$ The Number of Merchants.
12×65	13	$65a = 325$. The Number of Pounds each put in

Question 30. *Three Merchants join Stocks together; the first Man's Stock was less than the second Man's by* 13 l. *The second and third Man's Stock was* 175 l. *in trading they gain* 48 l. *more than their whole Stock was; the first Man's proportional Part of the Gain was* 78 l. *What was each Man's Stock and Part of the Gain?*

Let a, e, y represent each *Man's Stock*.

Then	1	$a + e + y = s$ The whole Stock.
	2	$s + 48 =$ The whole Gain.
And	3	$a + 13 = e$ By the *Question*.
	4	$e + y = 175$
$4 + a$	5	$a + e + y = 175 + a$
1, 5	6	$s = 175 + a$

6, 2

6, 2	7	$s+48=223+a$
But	8	$175+a:223+a::a:78$ *Per Queſt.*
8 ⁘	9	$aa+223a=78a+13650$
9 — 78*a*	10	$aa+145a=13650$
10 C☐	11	$aa+145a+5256,25=18906,25$
11 *ω* 2	12	$a+72,5=\sqrt{18906,25}=137,5$
12 — 72,5	13	$a=137,5-72,5=65$
3,	14	$e=a+13=78$
4 — 14	15	$y=97$
Then	16	$65:78::78:93\,l.\ 12\,s.=e's$ — — — Gain.
Again	17	$65:78::97:116\,l.\ 8\,s.=y's$ — — — Gain.
Proof	18	$116\,l.\ 8\,s.+93\,l.\ 12\,s.+78\,l.=288\,l.$ The Gain.
	19	$65+78+97=240.$ The whole Stock.
18 — 19	20	$288-240=48.$ The Gain more than the Stock.

Queſtion 31. *A Father at his Death left his three Sons all his Money in this Manner; to the Eldeſt he gave half of it, wanting 44 Pounds; to the Second he gave one third of it, and 14 Pounds more; to the Youngeſt he gave the Remainder, which was leſs than the Share of the ſecond Son by 82 Pounds; What was each Son's Share?*

Let a, e, y be the three Shares, and $z=$ the whole Sum.

Then	1	$a+e+y=z$
	2	$a=\frac{1}{2}z-44$
	3	$e=\frac{1}{3}z+14$ By the *Queſtion.*
	4	$y=\frac{1}{3}z+14-82$
2 + 3 + 4	5	$a+e+y=\frac{2z}{3}+\frac{z}{2}-98$
1, 5	6	$z=\frac{2z}{3}+\frac{z}{2}-98$
6 × 3	7	$3z=2z+\frac{3z}{2}-294$
7 × 2	8	$6z=4z+3z-588$
8 ±	9	$z=588.$ The whole Sum that was left.
2, 9	10	$a=\frac{588}{2}-44=250.$ The Eldeſt Son's Share.
3, 9	11	$e=\frac{588}{3}+14=210.$ The ſecond Son's, &c.
4, 9	12	$y=\frac{588}{3}+14-82=128.$ The Youngeſt &c.

Queſtion 32. *A Man playing at Hazard or Dice won the firſt Throw juſt ſo much Money as he had in his Pocket; the ſecond Throw he won the Square Root of what he then had and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was* 112 *l.* 16 *s.* *What Money had he when he began to play?*

Suppoſe

Suppose	1	$a =$ His first Sum.
$1 \times \bar{2}$	2	$2a =$ His Sum after the first Throw.
And	3	$5 + \sqrt{2a} =$ The Winnings at the 2d Throw.
$2 + 3$	4	$2a + 5 + \sqrt{2a} =$ The Sum after the 2d Throw.
4 Ⓢ 2	5	$4aa + 22a + 25 + 4a\sqrt{2a} : + 10\sqrt{2a}$ The Winnings at the 3d Throw; and therefore
$4 + 5$	6	$4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256$ *Sh.*

But to avoid these Surd Quantities. let us instead of supposing $a =$ the first Sum, make a second Trial.

Viz. let	1	$2aa =$ The first Sum.
$1 \times \bar{2}$	2	$4aa =$ The Sum after the first Throw.
Then	3	$2a + 5 =$ The Sum won at the second Throw.
$2 + 3$	4	$4aa + 2a + 5 =$ His Sum after the 2d Throw.
4 Ⓢ 2	5	$16a^4 + 16a^3 + 44aa + 20a + 25 =$ The Winnings at the 3d Throw; And therefore
$4 + 5$	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ *Shillings.*

Yet again, to avoid these high Æquations, let us make a third Supposition; Thus,

The	1	$\frac{aa}{2} =$ The first Sum.
$1 \times \bar{2}$	2	$aa =$ The Sum after the first Throw.
Then	3	$a + 5 =$ The Winnings at the 2d Throw.
$2 + 3$	4	$aa + a + 5 =$ The Sum after the 2d Throw.
Substi.	5	$e = aa + a + 5$
5 Ⓢ 2	6	$ee =$ The Winnings at the 3d Throw. Then
$5 + 6$	7	$ee + e = 2256$ Shillings by the Question.
7 C□	8	$ee + e + 0{,}25 = 2256{,}25$
8 ɯ 2	9	$e + 0{,}5 = \sqrt{2256{,}25} = 47{,}5$
$9 - 0{,}5$	10	$e = 47$
$5,\ 10$	11	$aa + a + 5 = 47$
$11 - \bar{5}$	12	$aa + a = 42$
12 C□	13	$aa + a + 0{,}25 = 42{,}25$
1 ɯ $\bar{2}$	14	$a + 0{,}5 = \sqrt{42{,}25} = 6{,}5$
$14 - 0{,}5$	15	$a = 6$
15 Ⓢ 2	16	$aa = 36$
$16 \div \bar{2}$	17	$\frac{aa}{2} = \frac{36}{2} = 18$ The Shill. he had in his Pocket when he began to play.

Note, In resolving of this Question, I have made three different Suppositions for the Thing sought; purely as an Instance to shew the young Learner, how well he ought to consider the Nature of the Question, when he first States it, and make choice of

representing the Thing sought, so as to avoid running it into Surds if possible, *viz.* as in the first Supposition of $a=$ the first Sum, *&c.* Not but that such Æquations may be solved, as shall be shewed in the next Chapter. However, it is most like an Artist to perform Things of this Nature the nearest and easiest Way they can be done.

Question 33. *Suppose there were two equal Circles, whose Peripheries* (viz. *Circumferences*) *are divided into* 44310 *equal Parts; and that those Circles were so placed upon one Axis, as to move the contrary Way to each other; and suppose one of them to move, but one of those equal Parts the first Day, two Parts the second Day, three Parts the third Day, and so on in Arithmetical Progression,* viz. 1, 2, 3, 4, 5, &c. *And the other to move every Day the Cubes of those Parts,* viz. 1. 8. 27. 64. 125, &c. *of the same Parts. How many Parts, and how many Days must each Circle move, before the same two Points meet that were together when they began to move?*

In order to give a ready Solution to this Question (or any other in this Kind) it will be convenient to premise this *Lemma.*

Lemma.

The *Sum* of any *Series* of *Cubes* whose *Roots* are in *Arithmetical Progression* (*the first Term, and common Difference being Unity or* 1) is equal to the *Square* of the *Sum* of all those *Roots.*

As in these,

Terms in *Arith.* &c.	Their *Cubes.*
1	1
2	8
3	27
4	64
5	125
6	216, &c.

21 × 21 = 441 Sum of their Cubes.

Let	1	$a=$ The Sum of all the Parts the *1st Circle* moves.
Then	2	$aa=$ The Sum of all the Parts the *2d* moves,
Consequen.	3	$aa+a=44310$ By the Quest. (*per Lem.*
2 C□	4	$aa+a+0,25=44310,25.$

4 *w* 2

4 ω 2	5	$a + 0,5 = \sqrt{44310,25} = 210,5$
5 — 0,5	6	$a = 210$ } the Number of Parts the first Circle must move.
6 ⊛ 2	7	$aa = 44100$ } The Number of Parts the 2d Circle moves.

Next to find the Number of Days they moved, there is given, the first Term $= 1$, the common Difference $= 1$. And the Sum of all the Terms $= 210$, thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let $a = 1$ the first Terms, $e = 1$ the common Difference, and $s = 210$ the Sum of all the Terms, to find $y =$ the last Term. As *per Sect.* 1. *Chap.* 6.

Then $yy + ey = 2s + aa - ae$ by the 16th Step, *Page* 186.

That is, $yy + y = 210 \times 2 = 420$, &c.

Hence $y = 20$ the Number of Days required.

I shall now proceed to give an Example or two of the Method used in arguing about unlimited Questions, *viz.* such Questions which admit of various Answers, such as those in *Alligation Alternate*, promised in Page 117.

In order to shorten that Work, it will be *convenient* for the *Learner* to know the *two Signs* of *Comparison* $>$ And $<$. The *Sign* $>$ is of **Greater than**, As $b > a$ *signifies* that b is *Greater than a*. The *Sign* $<$ is of **Lesser than**. As $b < d$ *signifies* that b is *Lesser than d*, &c.

Example 1.

Question 34. *A Tobacconist hath three Sorts of Tobacco,* viz. *one of* 2 s. 8 d. *the Pound, another of* 20 d. *the Pound, and a third Sort of* 16 d. *the Pound; of these he would make a Mixture to contain* 56 *Pound that may be sold for* 22 d. *the Pound: How much of each Sort may he take?*

Let $a =$ the Quantity of that worth 32 Pence the Pound, $e =$ that of 20 Pence the Pound, And $y =$ that of 16 Pence the Pound;

Then $a + e + y = 56$
And $32a + 20e + 16y = 1232$ } *viz.* each Quantity multiplied into its own Price, equals their Sum multiplied into the mean Price.

This

This Queſtion being thus ſtated, it appears by *Rule* 1. *Page* 176. that it is capable of innumerable Anſwers; becauſe for any one of theſe three Letters $a \cdot e \cdot y$. there may be taken any Number at Pleaſure, provided it be leſs than 56. But although that may be truly done, yet there are ſeveral Ways of arguing about theſe Sorts of Queſtions, which will limit or bound them to all their proper or poſſible Anſwers in whole Numbers. Thus,

Let	1	$a + e + y = 56$	As above.
And	2	$32a + 20e + 16y = 1232$	
$1 - a$	3	$e + y = 56 - a$	
$2 - 32a$	4	$20e + 16y = 1232 - 32a$	
3×16	5	$16e + 16y = 896 - 16a$	
$4 - 5$	6	$4e = 336 - 16a$	
$6 \div 4$	7	$e = 84 - 4a$ Hence $a < \frac{84}{4} = 21$	
$3 - 7$	8	$y = 3a - 28$ Hence $a > \frac{28}{3} = 9\frac{1}{3}$	

From the two laſt Steps it appears, that the Quantity ſignified by a, ought to be leſs than 21, and Greater than $9\frac{1}{3}$; That is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of a. Conſequently there may be Eleven Anſwers to this Queſtion in whole Numbers.

Suppoſe $a = 10$ Then $e = 84 - 40 = 44$ *Per* 7th Step.
And $y = 30 - 28 = 2$ *Per* 8th Step.
Again, if $a = 11$ Then $e = 84 - 44 = 40$ *Per* 7th Step.
And $y = 33 - 28 = 5$ *Per* 8th Step. And ſo on for the Reſt, which will be as in the following Table.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be eaſy to find out and collect all the limited Anſwers to any Queſtion (of this kind) wherein there are only three Quantities propos'd to be mix'd: But when there are more than Three, then the Work requires a little more Trouble; becauſe the ſingle Limits of all the Quantities above Two muſt be found. That is, if there are four Quantities concern'd in the Queſtion, the Limits of two of them muſt be found; If five Quantities are concern'd, then the Limits of three of them muſt be found, *&c.* As in the following Queſtion.

Queſtion

Question 35. *Suppose it were required to mix four Sorts of Wines together; viz. one Sort worth* 7 s. 4 d. *the Gallon; another Sort worth* 4 s. 7 d. *the Gallon; a third Sort worth* 3 s. 8 d. *the Gallon; and a fourth Sort worth* 2 s. 9 d. *the Gallon; How much of each Sort may be taken to make a Mixture of* 63 *Gallons, so as that the whole Quantity may be sold for* 5 s. 6 d. *the Gallon; without Loss,* &c.

First let all these several Rates, and the mean Rate, be reduced to one Denomination, *viz.* into Pence.

Viz. $\left\{\begin{array}{ll} 7\,s.\ 4\,d. = 88\,d. & 4\,s.\ 7\,d. = 55\,d. \\ 3\,s.\ 8\,d. = 44\,d. & 2\,s.\ 9\,d. = 33\,d. \end{array}\right\}$ And $5\,s.\ 6\,d. = 66\,d.$

Then put $a =$ the Quantity of that worth 88 *d.* the Gallon; $e =$ that of 55 *d.* the Gallon; $y =$ that of 44 *d.* the Gallon; and $u =$ that of 33 *d.* the Gallon.

Then	1	$a + e + y + u = 63$ By the Question.
And	2	$88a + 55e + 44y + 33u = 4158 = 63 \times 66.$
$1 - a$	3	$e + y + u = 63 - a$
$2 - 88a$	4	$55e + 44y + 33u = 4158 - 88a$
3×33	5	$33e + 33y + 33u = 2079 - 33a.$
$4 - 5$	6	$22e + 11y = 2079 - 55a$
$6 \div 11$	7	$2e + y = 189 - 5a$ Hence $a < \frac{189}{5} = 37\frac{4}{5}$
3×55	8	$55e + 55y + 55u = 3465 - 55a$
$8 - 4$	9	$11y + 22u = 33a - 693$
$9 \div 11$	10	$y + 2u = 3a - 63$ Hence $a > \frac{63}{3} = 21$

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by a must be less than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: That is, it may be $a =$ any Number of Gallons betwixt 21 and $37\frac{4}{5}$.

Whence it follows, that there may be collected 16 Answers to this Question from the Limits of a only.

Next to find the Limits of e, y, and u.

Suppose	11	$a = 22$ Then will $5a = 110.$ And $3a = 66$
But	12	$2e + y = 189 - 5a = 79$ *Per* 7th Step.
$12 - 2e$	13	$y = 79 - 2e$ Hence $e < \frac{79}{2} = 39\frac{1}{2}$
Again	14	$e + y + u = 63 - a = 41$ *Per* third Step.
$14 - e$	15	$y + u = 41 - e$
$15 - 13$	16	$u = e - 38$ Hence $e < 38$

From

From the 13th and 16th Steps it appears, that if $a = 22$ Then $e = 39$. $y = 79 - 2e - 1$. And $u = e - 28 = 1$.

Again.

Suppose	17	$a = 23$ Then $5a = 115$. And $3a = 69$
But	18	$2e + y = 189 - 5a = 74$. *Per* 7th Step.
$18 - 2e$	19	$y = 74 - 2e$ Hence $e < \frac{74}{2} = 37$
Again	20	$e + y + u = 63 - a = 40$. *Per* 3d Step.
$20 - e$	21	$y + u = 40 - e$
$21 - 19$	22	$u = e - 34$. Hence $e > 34$.

From the 19th and 22d Steps it appears, that if $a = 23$. Then e may be either 35 or 36.

Once more for a further *Illustration.*

Let	23	$a = 24$ Then $5a = 120$. And $3a = 72$
But	24	$2e + y = 189 - 5a = 69$. *Per* 7th Step.
$24 - 2e$	25	$y = 69 - 2e$. Hence $e < \frac{69}{2} = 34\frac{1}{2}$
Again	26	$e + y + u = 63 - a = 39$ *Per* 3d Step.
$26 - e$	27	$y + u = 39 - e$
$27 - 25$	28	$u = e - 30$ Hence $e > 30$.

From hence it appears, that if $a = 24$, Then e may be either 31 . 32 . 33. Or 34. *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th, and 28th Steps, from whence the Values of y and u may be easily found.

$$\text{That is, if } \begin{cases} e = 31. & \text{Then } y = 7. & \text{And } u = 1 \\ e = 32. & y = 5. & u = 2 \\ e = 33. & y = 3. & u = 3 \\ e = 34. & y = 1. & u = 4 \end{cases}$$

Proceeding on in this manner with all the other single Values of a, there may be found above 120 Answers to this Question in whole Numbers: And if you please to put $a =$ Fractions, there may be found an indefinite Number of Answers; whereas the Rule of Alligation in Vulgar Arithmetick affords but only one Answer in Fractions; to wit, that of $a = 31\frac{1}{2}$. $e = 10\frac{1}{2}$. $y = 10\frac{1}{2}$. $u = 10\frac{1}{2}$. As may be easily try'd, *per Rule Page* 115, *&c.*

These two Examples being well understood (especially if the last be thorowly pursu'd) may suffice to shew the Method of limiting the Answers to all Sorts of Questions of this kind, I shall therefore conclude this Chapter of Questions, with giving a Solution to the Ænigma (or Riddle) proposed (but not answered) by Mr. *John Kersey,* in the Close of the *Appendix* to his *Arithmetick,* which affords several pretty Questions, the Solution whereof will discover a certain Sentence consisting of three Words, which must be found by the

the Help of Figures placed (or ſuppoſed to be placed) over the Twenty Four Letters of the *Alphabet*.

Thus $\begin{cases} 1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,. \text{ \&c.} & \text{Called } \textit{Indices.} \\ a\,.\,b\,.\,c\,.\,d\,.\,e\,.\,f\,.\,g\,. \text{ \&c.} & \text{to the laſt } \textit{Letter.} \end{cases}$

So that if the *Index* of any Letter be once found, the Letter to which it belongs, is conſequently known.

The Enigma.

1. If the Difference between the Indices of the ſecond Letter of the ſecond Word; and the third Letter of the firſt Word, be multiplied into the Difference of their Squares, the Product will be 576. And if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the *Index* of the ſaid third Letter being the Greateſt.

Let	1	$a =$ The Greater *Index*, or that of the 3d Letter.
And	2	$e =$ The Leſſer, or that of the 2d Letter.
Then	3	$\overline{a-e}\times\overline{aa-ee} = 576$ By the Queſtion.
	4	$\overline{a+e}\times\overline{aa+ee} = 2336$ By the Queſtion.
$3\times$	5	$aaa-aae-aee+eee = 576$
$4\times$	6	$aaa+aae+aee+eee = 2336$
$6-5$	7	$2aae+2aee = 1760$
$6+7$	8	$aaa+3aae+3aee+eee = 4096$
$8\ \omega\ 3$	9	$a+e = \sqrt[3]{4096} = 16$
$4 \div a+e$	10	$aa+ee = \frac{2336}{a+e} = \frac{2336}{16} = 146$
9 Ⓖ 2	11	$aa+2ae+ee = 256$
$11-10$	12	$2ae = 110$
$10-12$	13	$aa-2ae+ee = 36$
$13\ \omega\ 2$	14	$a-e = \sqrt{36} = 6$
$9+14$	15	$2a = 22$ From hence it appears, that the 3d Letter of
$15 \div 2$	16	$a = 11$ the 1ſt Word is *l*, and the 2d Letter of the
$9-16$	17	$e = 5$ 2d Word is *e*.

Note, *In order to ſet down the Letters (as they become found) in their proper Places, it may be convenient to ſupply the vacant Places with Stars.*

Thus { First Word. * * *l* * *   Second Word. * * *e* * * *   Third Word. * * * * *

2. The

2. The Indices laſt found, are the two Extremes of Four Numbers in Arithmetical Progreſſion, the leſſer Mean being the *Index* of the firſt Letter of the third Word; and the greater Mean is the *Index* of the fourth and laſt Letter of the firſt Word.

Viz. 5 . 7 . 9 . 11 are the four Terms in *Arith. Progreſſion.* Whence it appears, that *G* (whoſe *Index* is 7) is the firſt Letter of the third Word; and that *i* (whoſe *Index* is 9) is the fourth or laſt Letter of the firſt Word; which being placed down, will ſtand thus, ***li* **e****. *G****.

3. The ſecond Letter of the third Word, is the ſame with the third Letter of the firſt Word; and the fifth Letter of the third Word is the ſame with the laſt Letter of the firſt Word.

Whence the Letters will ſtand thus, ***li*. **e****. *Gl****i**.

4. The Sum of the Squares of the Indices of the firſt and ſecond Letters of the firſt Word is 520. And the Product of the ſame Indices is ſeven Ninths of the Square of the greater *Index*, which is the *Index* of the ſaid firſt Letter.

Let $a =$ the Greater, and $e =$ the leſſer *Index*.

Then	1	$aa + ee = 520$	According to the *Data*.
And	2	$ae = \frac{7}{9}aa$	
$2 - a$	3	$e = \frac{7}{9}a$	
$3 \circledS 2$	4	$ee = \frac{49}{81}aa$	
$1 - 4$	5	$aa = 520 - \frac{49}{81}aa$	
5×81	6	$81aa = 42120 - 49aa$	
$6 + 49aa$	7	$130aa = 42120$	
$7 \div 130$	8	$aa = \frac{42120}{130} = 324$	
$8\ ɯ\ 2$	9	$a = \sqrt{324} = 18$	It's Letter is *s*:
$3, 9$	10	$e = \frac{7}{9}a = 14.$	It's Letter is *o*.

Hence the Letter will ſtand thus, *Soli*. **e****. *Gl*.***i**.

5. The Difference between the two laſt *Indices*, is the *Index* of the firſt Letter of the 2d Word; *viz.* $18 - 14 = 4$ being the *Index* of the Letter *D*.

Then the Letters will ſtand thus, *Soli*. *De****. *Gl*.***i**.

6. The third and laſt Letter of the ſecond Word, alſo the third Letter of the third Word, are the ſame with the ſecond Letter of the firſt Word.

Hence

Hence the Letters will ſtand thus, *Soli Deo Glo * i **.

7. The *Sum* of the *Indices* of the fourth Letter of the third Word, and the ſixth or laſt Letter of the ſame Word, being added to their *Product* is 35. And the Difference of their Squares is 288. The *Index* of the laſt Letter being the leaſt.

Put $a =$ the Greater, and $e =$ the Leſſer *Index*, as before.

Then	1	$ae+a+e=35$ By the *Data*.
And	2	$aa-ee=288$
$1-a$	3	$ae+e=35-a$
$3\div a+1$	4	$e=\frac{35-a}{a+1}$ For $e\times a+1=ae+e$
$4 \circledS 2$	5	$ee=\frac{1225-70a+aa}{aa+2a+1}$
$2+5$	6	$aa=288+\frac{1225-70a+aa}{aa+2a+1}$
$6\times aa$ &c.	7	$a^4+2a^3+aa=288aa+576a+288+1225-70a+aa$
$7\pm$	8	$a^4+2a^3-288aa-506a=1513$

This laſt Æquation being reſolved according to the Method which ſhall be ſhewed in the next Chapter, it will be $a=17$. *It's Letter r; And from the 4th Step* $=\frac{35-a}{a+1}=1$ *the Index of the Letter a.*

Then theſe two Letters being placed according to the *Data* above, are all that are required by the *Ænigma* to complete theſe Words,

Soli Deo Gloria.

CHAP. X.

The *Solution* of Adfected Equations in Numbers.

Before we proceed to the Solution of *Adfected Æquations*, it may not be amiſs to ſhew the Inveſtigation (or Invention) of thoſe *Theorems* or *Rules* for extracting the Roots of ſimple Powers, made uſe of in Chapter 11. Part I.

I ſhall here make Choice of the ſame Letters, to repreſent the Numbers both given and ſought; as in my Compendium of *Algebra*.

Viz. Let
- G, always denote the given *Reſolvend*.
- $r =$ any *Number* taken as near the true *Root* as may be, whether it be *greater* or *leſs*.
- $e =$ the unknown Part of the *Root* ſought, by which r is to be either *increaſed* or *decreaſed*.

Then if r be any *Number leſs* than the true *Root* it will be $r + e =$ the *Root* ſought.

But if r be taken *Greater* than the true *Root* it will then be $r - e =$ the *Root* ſought.

And put D for the *Dividend* that is *produced* from G, after it is *leſſen'd* and *Divided* by r &c. (*into the Co-efficients of Adfected Æquations*) according as the *Nature* of the *Root* requires.

Theſe Things being premiſed, we may proceed to raiſing the *Theorems*.

Section I.

I. For the *Square Root*, viz. $aa = G$. Quere a.

Let	1	$r + e = a$
1 ⓢ 2	2	$rr + 2re + ee = aa = G$
$2 - rr$	3	$2re + ee = G - rr$ Call it D, *viz.* $D = G - rr$
Then	4	$\dfrac{D}{2r + e} = e$ This ſhews the 1ſt Method of extracting the *Square Root, Sect.* 5. *Ch.* 11. *Part* 1.
$3 \div 2$	5	$re + \frac{1}{2}ee = \dfrac{G - rr}{2} = D$

Which gives this *Theorem* $\dfrac{D}{r + \frac{1}{2}e} = e$

The

The Arithmetical Operations of both these *Theorems* you have in the *Examples* of *Section* 2. *Page* 126; To which I refer the Learner, supposing him by this *time* to understand them without any more Words than what is there exprest.

II. To *extract* the Cube *Root*; *viz.* $aaa = G$. Quere a.

Let	1	$r + e = a$ Supposing r less than the true *Root*.
1 ⓖ 3	2	$rrr + 3rre + 3ree + eee = aaa = G$
2 — rrr	3	$3rre + 3ree + eee = G - rrr$
3 ÷ $3r$	4	$re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D$

Let $\frac{eee}{3r}$ be rejected or cast off, as being of small Value:

Then it will be, $re + ee = D$, which gives this following

$$\text{Theorem} \left\{ \frac{D}{r + e} = e \right.$$

By this *Theorem* or *Rule*, the 1st and 2d *Examples* in *Case* 1. *Page* 132. are perform'd; the which being compared with this *Theorem* may be very easily understood.

Again, Suppose $aaa = G$, (as before) And let r be taken greater than the true Root.

Then	1	$r - e = a$
1 ⓖ 3	2	$rrr - 3rre + 3ree = a^3 = G$ } eee being rejected as above.
2 ±	3	$3rre - 3ree = rrr - G$
3 ÷ $3r$	4	$re - ee = \frac{rrr - G}{3r} = D$

Which gives this Theorem $\left\{ \frac{D}{r - e} = e \right.$

By this *Theorem* the third *Example* in *Case* 2. *Page* 133. is perform'd.

III. To *extract* the Biquadrate *Root*; viz. $a^4 = G$. Quere a.

Let	1	$r + e = a$ Supposing r *Less* than *Just*.
1 ⓖ 4	2	$r^4 + 4rrre + 6rree = a^4 = G$ } Rejecting all the Powers of e as above
2 — r^4	3	$4rrre + 6rree = G - r^4$
3 ÷ $2rr$	4	$2re + 3ee = \frac{G - r^4}{2rr} = D$

Which gives this Theorem $\left\{ \frac{D}{2r + 3e} = e \right.$

By

By this *Theorem* the *Biquadratc Root* of any *Number* may be extracted. But as I have already said, Page 134. those Extractions may be very well perform'd, by two Extractions of the *Square Root. Vide Example Page* 135.

IV. To *extract* the **Sursolid** *Root*, viz. $a^5 = G$. Quere a.

If r be taken less than just, then $r + e = a$. As before.

And $\Big\{\dfrac{G - r^5}{5r^3} = D$. Which gives this **Theorem** $\Big\{\dfrac{D}{r + 2e} = e$

By this *Theor.* the *Sursolid Root Examp.* 1. *Page* 136, is extracted. But if r be taken greater than just; Then $r - e = a$.

And $\Big\{\dfrac{r^5 - G}{5r^3} = D$. Which gives this **Theorem** $\Big\{\dfrac{D}{r - 2e} = e$

By this last *Theorem* the *Example* in *Page* 137 is perform'd.

I presume it needless to pursue the raising of these Theorems, for extracting the Roots of simple Powers any further; because the Method of doing it is general, how high soever they are; and therefore it may be easily understood by what is already done.

Notwithstanding I have already shewed the Solution of Quadratic Æquations two several Ways, *viz.* by casting off the lowest Term: And by completing the Square, *Vide Section* 2. *Page* 195 *&c.* Yet it may not be amiss to shew, how those Æquations may be resolved into Numbers by this *Universal* Method of continued Series, wherein if the first r be taken equal to the first true Root or single Side of the Resolvend; And every single Value of e (as it becomes found) be still added to it, for a new r. Then those Roots may be extracted without repeating a second Operation, As before in the single Powers.

Case 1. Let $aa + 2ba = G$. 'Tis required to find the Value of a.

Put		
	1	$r + e = a$
1 ⓦ 2	2	$rr + 2re + ee = aa$
$1 \times 2b$	3	$2br + 2be = 2ba$
$2 + 3$	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G$
$4 - rr$ &c.	5	$2re + 2be + ee = G - rr - 2br$
$5 \div 2$	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$

Which gives this **Theorem** $\Big\{\dfrac{D}{r + b + \frac{1}{2}e} = e$

Suppose

Suppose $b = 364$ And $G = 38692865$
If $r = 6000$ Then $rr = 36000000$ And $2br = 4368000$
But $36000000 + 4368000 = 40368000 > 38692865 = G$
Therefore the First $r < 6000$. Let $r = 5000$ Then

1st $r = 5000$	$19346432{,}5 = \frac{1}{2}G$		
$b = 364$	$-1432000, = \frac{1}{2}rr + br$		
1st $r + b = 5364$	$5026432{,}5 = D$	$(800 = e$	
$+\frac{1}{2}e = 400$	46112		
1 Divisor $5764)$	41523	$(60 = e$	
2d $r + b = 6164$	37164		
$+\frac{1}{2}e = 30$			
2 Divisor $6194)$	4359	$(7 = e$	
3d $r + b = 6224$	$43592{,}5$	$867 = e$	
$+\frac{1}{2}e = 3{,}5$	(0)		
3 Divisor $6227{,}5$			

$\left.\begin{array}{l}\text{First } r = 5000\\ +e = 867\end{array}\right\} = 5867 = a$ As was required.

Case 2. If $aa - 2ba = G$ Then proceeding as above, there will arise this *Theorem* $\left\{\dfrac{D}{r - b + \frac{1}{2}e} = e.\right.$ &c.

And in *Case* 3. *viz.* $2ba - aa = G$ you will have this *Theorem* $\left\{\dfrac{D}{b - r - \frac{1}{2}e}\right.$ &c. As above.

I think it needless to trouble the Reader with the Work of these two *Theorems* in Numbers; because if the last *Example* of *Case* 1. be understood, the other will be easy. Not but that the Method of completing the Square is very ready and easy, as you may observe by the Work in several Questions of this Chapter.

Section 3.

In the Solution of all Adfected Æquations, that are above (or higher than) Quadratics, it will be the best way to take $r =$ the next nearest Root of the Æquation: And then it will be $r + e = a$ if r be less than just; Or $r - e = a$ if r be greater than just (as at the Beginning of this *Chap.*)

And all the Powers of the unknown Part of the Root (*viz.* e) above its Square (ee) are to be rejected or cast off; As before in

in raiſing the *Theorems* for the ſimple Powers. And therefore it is, that to ſupply the Want of thoſe Powers (above ee in the *Theorem*) the Operation muſt be repeated: As in the *Example* of extracting the *Cube Root*. Page 133. *viz*. when the Figures in the *Root* conſiſt of more than three Places, (*Vide* Page 140, and 141.)

Suppoſe $aaa + ba = G$. Quere a.

Let	1	$r + e = a$ *viz*. Let r be ſuppoſed *Leſs* than *juſt*.
$1 \circledast 3$	2	$rrr + 3rre + 3ree = aaa$
$1 \times b$	3	$br + be = ba$
$2 + 3$	4	$rrr + br + 3rre + be + 3ree = a^3 + ba = G$
$4 \div 3r$	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$
$5 -$ &c.	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$

Which gives this *Theorem* $\left\{ \dfrac{D}{r + \dfrac{b}{3r} + e} = e \right.$

But if r be taken greater than *juſt*, Then it will be $re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D$ Which produces this

Theorem $\left\{ \dfrac{D}{r + \dfrac{b}{3r} - e} = e \right.$

By either of theſe two *Theorems* the Value of a may be eaſily found. Or rather otherwiſe as in the following Example.

Let $aaa + 24a = 587914$ Here $b = 24$.
Suppoſe the Firſt $r = 90$ Then $r^3 = 729000 > 587914$ without the 24×90 being added to it: Therefore $r < 90$ Again, Suppoſe $r = 80$ Then $r^3 = 512000$ And $24r = 1920$ But $512000 + 1920 = 519320 < 587914$ Hence > 80, but nearer to it than 90. Therefore

it muſt be	1	$r + e = a$ *Leſs* than *juſt*
$1 \circledast 3$	2	$rrr + 3rre + 3ree = aaa$
1×24	3	$24r + 24e = 24a$
2, in Numb.	4	$512000 + 19200e + 240ee = aaa$
3, in Numb.	5	$1920 + 24e = 24a$
$4 + 5$	6	$513920 + 19224e + 240ee = 587914$
$6 - 513920$	7	$19224e + 240ee = 73994$
$7 \div 240$	8	$80{,}1e + ee = 308{,}31 = D$
$8 \div$	9	$e = \frac{D}{80{,}1 + e}$

Operation

Operation 80,1)	308,31 = D	(3,7, = e
$+e =$,3	249 3	
1. *Divisor* 83,1)	59,01	First $r = 80,$
$+e =$,7	58.66	$+e =$ 3,7
2. *Divisor* 83,8)	,35	$r+e = 83,7$

Or rather new $r = 83,7$ for a second Operation, which being involved and tryed (as above) will be found Greater than just; Therefore

it must be		
	1	$r - e = a$
1 ⓔ 3	2	$rrr - 3rre + 3ree = aaa$
1 × 24	3	$24r - 24e = 24a$
2, in Numb.	4	$586376,253 - 21017,07e + 251,1ee = aaa$
3, in Numb.	5	$2008,8 - 24e = 24a$
4 + 5	6	$588385,053 - 21041,07e + 251,1ee = 587914$
6 ±	7	$21041,07e - 251,1ee = 471,053$
7 ÷ 251,1	8	$83,7955e - ee = 1,87595778 = D$
8 ÷	9	$e = \dfrac{D}{83,7955 - e}$

2. *Operation* 83,7955)	1,87595778	(,0223 = e
$-e =$,02	1,675510	
1. *Divisor* 83,7755)	,2004477	
$-e =$,002	,1675470	
2. *Divisor* 83,7735)	,03290078	
$-e =$,0003	,02513196	
3. *Divisor* 83,7732)	,00776882	

Having once found half the Places of Figures for the Value of e, it will be needless to form New Divisors (as above); for the rest of the Figures may be as truly found by plain Division only. Thus

The last Divisor is 83,7732)	,007768820	(,0223 = e } Add
	7539588	(,0000927 }
Last $r = 83,7$	2292320	,0223927 = e
$-e =$,0223927	1675464	
$r - e = 83,6776073 = a$	6168560	
	5864124 &c.	

But if more Exactness be required, you may make the New $r = 83,6776073$ And proceed with it to a third Operation; which

which will afford Twenty Seven Places of Figures for the Value of a. That is, every Operation will produce triple the Places of Figures to thoſe of the Precedent r. And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be obſerved in the Solution of all Adfected Æquations (how high ſoever they are) according to this Method of reſolving them. *See Page* 141.

Example 2. Suppoſe $aaa - ba = G$. Quere a.

If $r + e = a$ Then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this *Theorem* $\left\{ \dfrac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e \right.$

But if $r - e = a$ Then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this *Theorem* $\left\{ \dfrac{D}{r + \frac{\frac{1}{3}b}{r} + e} = e \right.$

Or you may proceed otherwiſe, as in the laſt *Example.*

Let $aaa - 6438a = 104785688$ Here $b = 6438$
Suppoſe the Firſt $r = 500$. $rrr = 125000000$ and $br = 3219000$
Then $125000000 - 3219000 = 121781000$
But $121781000 > 104785688$ Therefore $r < 500$
Again, Suppoſe $r = 400$ $rrr = 64000000$. and $br = 2575200$
Then will $64000000 - 2575200 = 6142800$
But $61424800 < 104785688$ Hence $r > 400$

Conſequently r is betwixt 400 and 500. But 500 is the next neareſt; Therefore, Let $r = 500$ being Greater than juſt.

Then	1	$r - e = a$
1 ⓒ 3	2	$rrr - 3rre + 3ree = aaa$
$1 \times b$	3	$br - be = ba$
2, in Numb.	4	$125000000 - 750000e + 1500ee = aaa$
3, in Numb.	5	$3219000 - 6438e = 6438a$
$4 - 5$	6	$121781000 - 743562e + 1500ee = 104785688$
$6 \pm$	7	$743562e - 1500ee = 16995312$
$7 \div 1500$	8	$495e - ee = 11330 = D$
$8 \div$	9	$e = \dfrac{D}{495 - e}$

Operation

Operation	495)	11330	(23,8 = e		
$-e=$	20	950			
1. *Divisor*	475)	1830		First $r=$	500
$-e=$	3	1416		$-e=$	23,8
2. *Divisor*	472)	414,0		$r-e=$	476,2 = a
		377,6			

Let New $r=476$ for a 2d Operation. Then $r^3=107850176$ and $br=3064488$ But $107850176-3064488=104785688$ the same with the Resolvend. Consequently $a=476$ *just*.

Example 3. Let $ba-aaa=G$. Quere a.

If $r+e=a$ Then $\frac{\frac{1}{3}be}{r}-re-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{3}b=D$

which gives this *Theorem* $\left\{\dfrac{D}{\dfrac{\frac{1}{3}b}{r}-r-e}=e\right.$

But if $r-e=a$ Then $re-\frac{\frac{1}{3}be}{r}-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{3}b=D$

which gives this *Theorem* $\left\{\dfrac{D}{r-\dfrac{\frac{1}{3}b}{r}-e}=e\right.$

Or otherwise as before in the two last *Examples*. Thus

Let $123456a-aaa=12272861$. Here $b=123456$. Suppose the First $r=200$ Then $rrr=8000000$. and $br=24691200$. then $24691200-8000000=16691200$ but $16691200>12272861$: therefore r is here less than just, because the highest Power is $-$, or Negative.

Again, Suppose $r=300$ then $r^3=27000000$ and $br=37036800$ Then $37036800-27000000=10036800<12272861$ Consequently $r<300$ and $r>200$

Let $r=300$. being the next nearest, but more than just.

Then	1	$r-e=a$
1 ⓒ 3	2	$rrr-3rre+3ree=aaa$
$1\times b$	3	$br-be=ba$
2, in Numb.	4	$27000000-270000e+900ee$
3, in Numb.	5	$37036800-123456e$
$5-4$	6	$10036800+146544e-900ee=12272861$
$6-$	7	$146544e-900ee=2236061$
$7\div 900$	8	$162e-ee=2484=D$
$1\div$ &c.	9	$e=\dfrac{D}{162-e}$

Operation.	162)	$\dot{2}4\dot{8}4$	(16,6 = e	
$-e=$	10	152		
1. *Divisor*	152)	964	First $r=$	300
$-e=$	6	876	$-e=$	16,6
2. *Divisor*	146)	88,0	$r-e=$	$283{,}4=a$
		87,6		

Or New $r=283$ which being involved, &c. will appear to be the true Root. That is, $a=283$ just.

Note, These are usually called the three Forms of Cubic Æquations; and in the Solution of the third or last Form, *viz.* $ba-aaa=G$, you may meet with some seeming Difficulties; especially in making Choice of the First r because this Æquation is an ambiguous Æquation, and hath two Affirmative Roots, *viz.* a Greater and Lesser Root. But having once found either of them, the other may be easily obtained by Division only; As in the Quadratic Æquations *Vide Chapter* 8.

As for Instance, in the last Example, $a=283$
And $1234567a - aaa = 12272861$. Make these two Æquations $=0$. To wit, Let $a-283=0$.
And $-aaa+123456a-12272861=0$.

$$
\begin{array}{lll}
\text{Then, } a-283) & -aaa+123456a-12272861 & (-aa \\
 & -aaa+283aa & \\
\hline
 & -283aa+123456a & (-283a \\
 & -283aa+80089a & \\
\hline
 & +43367a-12272861 & (+43367 \\
 & +43367a-12272861 & \\
 & (0) \qquad (0) &
\end{array}
$$

Hence it appears that $-aa-283a+43367=0$
Consequently $aa+283a=43367$ this Æquation being Solved, $a=110{,}2722$ &c. which is the lesser Root of the aforesaid Æquation $ba-aaa=G$ &c.

After this manner all the possible, and impossible Roots of any Æquation may be easily discovered, any one of its Roots being once found. I shall therefore omit inserting more Examples of that Kind.

Suppose $aaa+baa+ca=G$. Quere a.
Let $b=74$, $c=8729$. and $G=560783$
By Trial (as before) it will be found that the next nearest $r=40$ being something less than just.

Therefore

Therefore	1	$r+e=a$
1 × c	2	$cr+ce=ca$
1 ⓖ 2 : × b	3	$brr+2bre+bee=baa$
1 ⓖ 3	4	$rrr+3rre+3ree=aaa$
2, in Numb.	5	$349160+8729e$
3, in Numb.	6	$118400+5920e+74ee$
4, in Numb.	7	$64000+4800e+120ee$
5 + 6 + 7	8	$531560+19449e+194ee=560783$
8 − 531560	9	$19449e+194ee=29223$
9 ÷ 194	10	$100{,}2e+ee=153{,}06=D$
10 ÷	11	$e=\dfrac{D}{100{,}2+e}$

Operation. 100,2) 153,06 (1,5 = e
$+e=$ 1 101,2 First $r=40$
$+e=$ 1,5

1. *Divisor* 101,2) 51,86
$+e=$,5 50,85 $r+e=41{,}5=a$

2. *Divisor* 101.7 1,01

Or New $r=41.5$ for a second Operation, which being duly involved, *&c.* will be found more than just.

Therefore	1	$r-e=a$
Then	2	$cr-ce=ca$
	3	$brr-2bre+bee=baa$
	4	$rrr-3rre+3ree=aaa$

These being turn'd into Numbers, *&c.* As above, they will be $20037{,}75e-198{,}5ee=390{,}375$ — — — which being Divided by 198,5 the Co-efficient of ee — — will become $100.946e-ee=1{,}966624$ &c. $=D$.

Operation. 100,946 1,966624 (,019 = e
$-e=$,01 1,00936

1. *Divisor* 100,936) 957264
$-e=$,009 908343

2. *Divisor* 100,927) * 489210 (,0004847
403708

,0194847 = e

* Here I proceed by plain Division, without forming New Divisors.

855020
807416

Last $r=41{,}5$ 476040
$-e=$,0194847 403708

$r-e=41{,}4805153=a$ 723320 &c.

Let

Let the laſt Æquations in the *Ænigma, Chap.* 9. be here propoſed for a Solution.

Viz. $aaaa + baaa - caa - da = G$

$b = 2, c = 288 \,.\, d = 506$: and $G = 1513$ Quere a.

By Tryals it will be found, that the next neareſt $r = 20$ being ſomething more than juſt.

Therefore		
1 × d	1	$r - e = a$
1 ☉ 2 : × c	2	$dr - de = da$
1 ☉ 3 : × b	3	$crr - 2cre + cee = caa$
1 ☉ 4	4	$brrr - 3brre + 3bree = baaa$
	5	$r^4 - 4rrre + 6rree = aaaa$

Theſe being turned into Numbers, and thoſe duly collected, according as the Signs of the Æquation direct, they will become

$50680 - 22374e + 2232ee = 1513$. which being all divided by 2232. the Co-efficient of ee, will become $10e - ee = 22 = D$.

Then $\left\{\dfrac{D}{10 - e} = e\right.$

Operation, $10)\ 22\ (3 = e$
$- e = 3 \quad 21$

Diviſor $7)\ 1$
Firſt $r = 20$
$- e = 3$

$r - e = 17 = a$ *juſt*. See the End of Chap. 9.

By what hath been already done, about the Solution of theſe few Æquations (being carefully obſerved,) I preſume the Learner will eaſily conceive how to proceed in the Solution of all Kinds of Æquations, be they never ſo high, or adfected; therefore I ſhall not here propoſe many various Examples, but only take them as they fall in Courſe when I come to the next Part, wherein you will (perhaps) find ſuch Æquations with their Solutions as are not common.

CHAP.

CHAP. XI.

Of *Simple* Interest, Annuities or *Pensions*, &c.

Interest or the Use paid for the *Loan* of *Money*, is either *Simple*; Or *Compound*.

Section 1. Of *Simple* Interest.

Simple Interest, is that which is paid for the Loan of any Principal or Sum of Money, lent out for some Time at any Rate *per Cent.* agreed on between the Borrower and the Lender; which according to the Laws of *England* ought to be six Pounds for the Use of 100 *l.* for one Year, and twelve Pounds for the Use of 100 *l.* for two Years. And so on for a greater, or lesser Sum, proportional to the Time proposed.

There are several Ways of computing (or answering Questions about) *Simple Interest*; as by the Single, and Double *Rule* of *Three* (*See Page* 96. &c.) others make use of Tables composed at several Rates *per Cent.* As Sir *Samuel Moreland* in his Doctrine of *Interest*, both Simple and Compound, which is all perform'd by Tables; wherein he hath detected several Material Errors committed by Doctor *Newton*, Mr. *Kersey* upon *Wingate* and Mr. *Clavil* &c. in the Business of Computing Interest, &c. by their Tables, too tedious to be here repeated.

But I shall in this Tract take other Methods, and shew that all Computations relating to Simple Interest, are grounded upon Arithmetical Progression; and from thence raise such General *Theorems*, as will suit with all Cases. In order to that,

Let $\left\{\begin{array}{l} P = \text{Any Principal or Sum put to Interest.} \\ R = \text{The Ratio of the Rate, } per\ Cent.\ per\ Annum. \\ t = \text{The Time of the Principal's Continuance at Interest.} \\ A = \text{The Amount of the Principal, and its Interest.} \end{array}\right.$

Note; the Ratio of the Rate, is only the Simple Interest of 1 *l.* for one Year, at any given Rate; and it is thus found.

Viz. 100 : 6 : : 1 : 0,06 = the Ratio at 6 *per Cent. per Ann.*
Or 100 : 7 : : 1 : 0,07 = the Ratio at 7 *per Cent.* &c.
Again 100 : 7,5 : : 1 : 0,075 = the Ratio at 7 and ½ *per Cent.*

And if the given Time be whole Years; then $t =$ the Number of those Years: But if the Time given, be either pure Parts of a Year, or Parts of a Year mix'd with Years; those Parts must be turn'd into Decimals; and then $t =$ those Decimals, &c. Now the Common

Parts

Parts of a Year may be eaſily turn'd or converted into Decimal Parts, if it be conſidered

That One $\begin{cases}\text{Day is the } \frac{1}{365} \text{ Parts of a Year} = 0{,}00274 \textit{ fere}\\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = 0{,}0833333 \text{ \&c.}\\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = 0{,}25\end{cases}$

Half a Year $= 0{,}5$ And three Quarters $= 0{,}75$

Theſe Things being premiſed, we may proceed to raiſing the *Theorems*.

Let $R =$ the Intereſt of 1 *l.* for one Year. As before.

Then $2R =$ the Intereſt of 1 *l.* for two Years.

And $3R =$ the Intereſt of 1 *l.* for three Years.

$4R =$ the Intereſt of 1 *l.* for four Years. And ſo on for any Number of Years propoſed.

Hence it is plain, that the Simple Intereſt of one Pound is a Series of Terms in Arithmetical Progreſſion increaſing; whoſe firſt Term and common Difference is R. And the Number of all the Terms is t. Therefore the laſt Term will always be $tR =$ the Intereſt of 1 *l.* for any given Term ſignified by t.

Then $\begin{cases}\text{As one Pound : Is to the Intereſt of 1 } l. \text{ :: So is any}\\ \text{Principal or given Sum : To its Intereſt.}\end{cases}$

That is, $1\,l. : tR :: P : tRP =$ the Intereſt of P. Then the Principal being added to its Intereſt, their Sum will be $= A$ the Amount required: Which gives this general *Theorem*.

Theorem $tRP + P = A$

From whence the three following *Theorems* are eaſily deduced,

Theorem 2. $\frac{A}{tR+1} = P$. *Theorem* 3. $\frac{A-P}{tP} = R$

Theorem 4. $\frac{A-P}{RP} = t$.

Theſe four *Theorems* reſolve all Queſtions about Simple Intereſt.

Queſtion 1. *What will* 256l. 10s. *Amount to in* 3 *Years,* 1 *Quarter,* 2 *Months, and* 18 *Days, at* 6 per Cent. per Ann.

Here is given $P = 256\ 5$. $R = 0{,}06$. And $t = 3{,}46599$

For 3 Years $= 3$, Quere A. *Per Theorem* 1.

one Quarter $= 0.25$

2 Months $= 0.16667 = 0{,}08333 \times 2$

18 Days $= 0{,}04932 = 0{,}00274 \times 18$

Hence $t = 3{,}46599 \times 0{,}06 = 0{,}2079594 = tR$

Then $0.2079594 \times 256{,}5 = 53\ 341586 = tRP$

And $53{,}341586 + 256{,}5 = 309{,}841586 = tRP + P = A$

That is, $309{,}841586 = 309\,l.\ 16\,s.\ 10\,d.$ being the Anſwer required.

Queſtion

15. 41 . 0 / 15 . 0 - 2 5
46 4 . 7 5 X

Question 2. *What Principal or Sum being put to Interest, will raise a Stock of* 309 l. 16 s. 10 d. in three Years, one Quarter, two Months and 18 Days; at 6 *per Cent. per Annum*?

Or the same Question otherwise stated thus.

What is 309 l. 16 s. 10 d. *due* 3 *Years, one Quarter,* 2 *Months and* 18 *Days hence, worth in ready Money; Abating or Discounting* 6 *per Cent.* &c.

Here is given $A = 309{,}841586$ $R = 0{,}06$ $t = 3{,}46599$ (found as before) Thence to find P. *Per Theorem* 2.

First $3{,}46599 \times 0{,}06 = 0{,}2079594 = tR$

Then $tR + 1 = 1{,}2079594)\ 309{,}841586 = A\ (256{,}5 = P$

That is, $256{,}5 = 256$ *l.* 10 *s.* the Answer required.

Question 3. *At what Rate of Interest, per Cent,* &c. *will* 256 l. 10 s. *Amount to* 309 l. 16 s. 10 d. *In* 3 *Years, one Quarter, two Months and* 18 *Days.*

Here is given, $P = 256{,}5$, $A = 309{,}841586$ and $t = 3{,}46599$

To find R. *Per Theorem* 3.

First $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$

Next $3{,}46599 \times 256{,}5 = 889{,}026435 = tR$

And $tR = 889{,}026435)\ 53{,}341586\ (00{,}06 =$ Ratio.

Then 1 *l.* : 0,06 :: 100 : 6 = the Rate required.

Question 4. *In what Time will* 256 l. 10 s. *raise a Stock of (or Amount to)* 309 l. 16 s. 10 d. *at* 6 *per Cent.* &c.

Here is given, $P = 256{,}5$ $A = 309{,}841586$ and $R = 0{,}06$

To find t *Per Theorem* 4.

First $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$

And $256{,}5 \times 0{,}06 = 15.39 = PR$

Then $15{,}39)\ 53{,}341586\ (3{,}46599 = t$

That is $t = 3$ Years and ,46599 Decimal Parts of a Year; which may be brought into Common Parts of a Year, thus

0,46599	And 0,08333) 0,21599 (2 *Months.*
—0,25 = *one Quarter*	,16666
0,21599	0,02074) ,04933 . (18 *Days.*

Hence $t = 3$ *Years, one Quarter,* 2 *Months, and* 18 *Days*; the Answer required.

It must needs be easy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Interest, by forming the Ratio, *viz.* R accordingly.

Scholium

Scholium.

Altho' it be according to the Laws and Custom of *England*, to compute Interest at the Proportion of *6 per Cent.* (as above) yet he that takes up Money at Interest for any Time less than even or complete Years, pays more Interest than seems reasonably due, according to the true Rules of Art.

As for Instance; If 100 *l.* be forborn at Interest one whole Year, it amounts to 106 *l.* But (I say) if it be paid at the half Year's End, it should not amount to 103 *l.* As appears from this following Proportion.

Let $a=$ the Amount due at the half Year's End; Then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo* $aa = 10600$ And $a = \sqrt{10600} = 102,9563 = 102$ *l.* 19 *s.* $1\frac{1}{2}$ *d.* which is less than 103 *l.* by $10\frac{1}{2}$ *d.* And if it be paid in less than half a Year's Time, the Error must needs be the Greater.

Section 2. Of Annuities or Pensions *in Arrears; Computed at Simple Interest.*

Annuities or *Pensions*, &c. are said to be in Arrears, when they are payable or due, either Yearly or Half-yearly, *&c.* and are unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became due: Now in order to that,

Put $\begin{cases} u = \text{the Annuity, Pension, or yearly Rent, \&c.} \\ t = \text{the Time of its Continuance, or being unpaid.} \\ R = \text{the Ratio, or Interest of 1 } l. \text{ for 1 Year,} \\ A = \text{the Amount of the Annuity and its Interest.} \end{cases}$ As before.

Then if $u=$ the first Year's Rent, due without Interest.

$Ru =$ the Interest, And $2u =$ the Rent. } Due at the End of the second Year.

$2Ru =$ the Interest, And $3u =$ the Rent. } Due at the End of the third Year.

$3Ru =$ the Interest, $4u =$ the Rent. } Due at the End of the fourth Year.

$4Ru =$ the Interest, $5u =$ the Rent. } Due at the End of the fifth Year.

And so on for any Number of Years. Hence it is evident, that $R + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the Rents and their Interest, being forborn 5 Years.

From

From whence it follows, that $Ru + 2Ru + 3Ru + 4Ru = A - tu$.

Here $t = 5$. Divide by u, Then $R + 2R + 3R + 4R = \frac{A - tu}{u}$

Next to find the Sum of this Progression (*See Page* 185.) Thus Let $R + 2R + 3R + 4R$ &c. $= z$. Then $1 + 2 + 3 + 4$ &c. $= \frac{z}{R}$

Here the Sum of the first and last Terms are $4 + 1 = 5 = t$
And the Numbers of all the Terms is $4 = t - 1$. Therefore $\frac{t-1}{2} \times t =$ the Sum of all the Terms. That is, $\frac{tt - t}{2} = \frac{z}{R}$

Hence $\frac{ttR - tR}{2} = z$. Consequently $\frac{ttR - tR}{2} = \frac{A - tu}{u}$

Now from this Æquation it will be easy to deduce the following *Theorems*.

Theorem 1. $\left\{\frac{ttRu - tRu + 2tu}{2}\right. = A$. Or $\left\{\frac{ttu - tu}{2}\right. \times R : + tu = A$

Theorem 2. $\left\{\frac{2A}{ttR - tR + 2t}\right. = u$. *Theorem* 3. $\left\{\frac{2A - 2tu}{ttu - tu}\right. = R :$

Let $\frac{2}{R} - 1 = x$. Then $t = \sqrt{\frac{2A}{Ru} + \frac{xx}{4}} : - \frac{1}{2}x \Big\}$ *Theorem* 4.

Question 1. *If* 250l. *Yearly Rent (or Pension, &c.) be forborn or unpaid Seven Years; What will it amount to in that Time, at* 6 per Cent. *for each Payment as it becomes due?*

Here is given $u = 250 . t = 7$. And $R = 0{,}06$ To find A. *Per Theor.* 1. First $250 \times 7 = 1750 = tu$. $1750 \times 7 = 12250 = ttu$
Again $12250 - 1750 = 10500 = ttu - tu$. And $\frac{10500}{2} \times 0{,}06 = 315$
Lastly $315 + 1750 = 2065 = A$. *Viz.* 2065 *l.* the *Answer* required.

But if the Annuity, Rent or Pension, is to be paid by Quarterly or Half-yearly Payments, &c.
Then $\frac{0{,}06}{2} = 0{,}03 = R$ for Half-yearly Payments.
And $\frac{0{,}06}{4} = 0{,}015 = R$ for Quarterly; Or $0{,}045 = R$ for Three-quarterly Payments. Example of Half-yearly Payments.

Suppose 250 *l.* *per Annum*, to be paid by Half-yearly Payments, were in Arrears or unpaid for seven Years; What would it amount to allowing *6 per Cent. per Annum* for each Payment as it became due.

In this Example there is given $u = 125 = \frac{250}{2}$ $t = 14$ the Number of Payments; And $R = 0{,}03 = \frac{0{,}06}{2}$. Thence to find A.

First $125 \times 14 = 1750 = tu$. $1750 \times 14 = 24500 = ttu$.
Again $24500 - 1750 = 22750 = ttu - tu$. Then $\frac{22750}{2} = 11375$

 And

And $11375 \times 0{,}03 = 341{,}25$ Laſtly $341{,}25 + 1750 = 2091{,}25$, that is, $A = 2091$ *l.* 5 *s.* the Anſwer required.

N. B. Hence it may be obſerved, that Half-yearly Payments are more advantageous than Yearly:

For 2091 *l.* 5 *s.* > 2065 *l.* by 26 *l.* 5 *s.* Conſequently, Quarterly Payments are more advantageous than Half-yearly Payments.

Queſtion 2. *What yearly Rent, Penſion,* &c. *being forborn or unpaid ſeven Years, will raiſe a Stock of* 2065 l. *allowing* 6 per Cent. per Annum *for each Payment as it becomes due?*

Here is given $A = 2065$. $t = 7$ And $R = 0{,}06$. To find u. *Per Theorem* 2.

Firſt $7 \times 0{,}06 = 0{,}42 = tR$, and $0{,}42 \times 7 = 2{,}94 = ttR$.

Then $ttR - tR = 2{,}52$

Laſtly $ttR - tR + 2t = 16{,}52)\ 4130 = 2A\ (250 = u$.

That is, 250 *l. per Annum*, &c. will raiſe 2065 *l.* the Stock required.

Queſtion 3. *In what Time will* 250 l. *yearly Rent, raiſe a Stock of* 2065 l. *allowing* 6 per Cent. *&c. for the Forbearance of the Payments as they become due?*

Here is given $u = 250$. $A = 2065$ And $R = 0{,}06$ To find t. *Per Theorem* 4. Firſt

$\frac{2}{R} = \frac{2}{0{,}06} = 33{,}3333$ And $33{,}3333 - 1 = 32{,}3333 = x = \frac{2}{R} - 1$

Then 16.16666 &c, $= \frac{1}{2}x$, $261{,}3605$ &c. $= \frac{1}{4}xx$.

Again $\frac{4130}{15} = 275{,}333 = 2A \div Ru$ And $275{,}3333 + 261{,}3605$

$= 536{,}6938 = \frac{2A}{Ru} + \frac{1}{4}xx$. Then $\sqrt{536{,}6938} = 23{,}1666$.

Laſtly $23{,}1666 - 16{,}1666 = 7 = t$ the Time required.

Queſtion 4. *If* 250 l. *yearly Rent, being forborn ſeven Years, will amount to* 2065 l. *allowing Simple Intereſt for every Payment as it becomes due, what muſt the Rate of the Intereſt be* per Cent. &c.

Here is given $u = 250$. $A = 2065$. And $t = 7$. To find R: *Per Theorem* 3.

$$\text{Thus}\begin{cases} ttu = 12250\ .\ 4130 = 2A \\ -\ tu -\ \ 1750\ .\ 3500 = 2tu \end{cases}$$

$ttu - tu = 10500)\ \ 630 = 2A - 2tu\ (0{,}06 = R$

Then $1 : 0{,}06 :: 100 : 6$ the Rate required.

Section

Section 3. *The* Present *Worth* of Annuities or *Pensions*, &c. *computed at Simple Interest.*

The Business of Purchasing Annuities, or taking of Leases, *&c.* for any assigned Time, depends upon the true Æquating of the Principal or Money laid out on the Purchase, with the Annuity or yearly Rent, by allowing (or discompting) the same Rate of Interest to both Parties. Which may be easily perform'd by duly applying the respective *Theorems* of the two last *Sections* together. As will fully appear by the following *Question.*

Question 1. *What is* 75l. *Yearly Rent, to continue nine Years, worth in ready Money,* at 6 per Cent. per Annum *Simple Interest* ?

1. *Per Theorem* 1. of the last *Section*, find what the proposed Yearly Rent would amount to, if it were forborn 9 Years at 6 *per Cent.*

Thus $u = 75 \cdot t = 9$. And $R = 0{,}06$ Quere A.

$ttu = 6075$

$tu = 675$

$ttu - tu = 5400$

Then 2) 5400 (2700 } Multiply
$R = 0{,}06$

162,
$+ tu = 675$ } $= 837 = A$

2. Then by *Theorem* 2. *Section* 1. find what Principal, being put to Interest for the same Time, and at the same Rate, will amount to 837 *l.* $= A$.

Thus $tR = 0{,}54 = 9 \times 0{,}06$. $tR + 1 = 1.54$) 837 (543,5064 $= P$

That is, $P = 543$ *l.* 10 *s.* $1\frac{1}{2}$ *d.* Which is the Worth of 75 *l.* a Year, as was required.

From the Work of these two Operations, (duly consider'd) it must needs be easy to conceive, how the two *Theorems* by which they were perform'd, may be combined into one.

For 1. $\frac{ttRu - tRu + 2tu}{2} = A$. And 2. $PtR + P = A$.

Consequently $PtR + P = \frac{ttRu - tRu + 2tu}{2}$ And from this Æquation may be deduced the following *Theorems*.

Theorem 1. $\left\{\frac{ttRu - tRu + 2tu}{2tR + 2}\right. = P$. Or $\frac{ttR - tR + 2t}{2tR + 2} : \times u = P$

By this *Theorem* all *Questions* of the same kind with the last (*viz.* that above may be easily and readily answered at one Operation.

Theorem 2. $\left\{\dfrac{2PtR + 2P}{ttR - tR + 2t} = u.\right.$ Or $\dfrac{tR + 1}{ttR - tR + 2t} : \times 2P = u.$

Theorem 3. $\left\{\dfrac{2P - 2tu}{ttu - tu - 2Pt} = R\right.$

Let $\dfrac{2}{R} - \dfrac{2P}{u} - 1 = x$. Then will $tt \pm xt = \dfrac{2P}{Ru}$

Which gives this *Theorem* 4. $\left\{\sqrt{\dfrac{2P}{Ru} + \dfrac{xx}{4}} : \pm \dfrac{x}{2} = t\right.$

By the ſecond and third *Theorems*, two very uſeful *Queſtions* may be eaſily anſwer'd.

1. *As for Inſtance: If it be required to find what Annuity, or yearly Rent,* &c. *may be purchaſed, for any propoſed Sum, to continue any aſſigned Time, allowing any Rate of Intereſt.*

This Queſtion may be anſwered by Theorem 2.

2. *Again: If it be required to find how long any yearly Rent, Penſion, or Annuity,* &c. *may be purchaſed (or enjoy'd) for any propoſed Sum, at any given Rate of Intereſt.*

All Queſtions of this Kind are eaſily anſwered *per* Theorem 4.

In theſe Queſtions it is ſuppoſed, that the Purchaſer or yearly Rent, is to commence or be immediately enter'd upon. But if it be required to find the Value or Purchaſe of any Annuity or yearly Rent, *&c.* in Reverſion; That is, when it is not to be enter'd upon until after ſome Time, or Number of Years are paſt; then you muſt firſt find what the Sum propos'd to be laid out in the Purchaſe, would amount to, if it were put to Intereſt during the Time the Annuity, *&c.* is not to be in preſent Poſſeſſion; and make that Amount the Sum for the Purchaſe proceeding with it as in either of the two laſt Queſtions, *&c.*

Note, *From the firſt Queſtion of this Section it will be eaſy to conceive how to perform the Æquation of Payments, between Debtor and Creditor, at any Rate of Intereſt, without doing any Damage to either Party.*

That is when ſeveral Sums of Money are to be paid, at ſeveral different Times, to find the Time when all the Payments may be truly diſcharged at once: As if one Sum were to be paid at the End of two Months, another at ſix Months, and perhaps a third Sum at eight Months End, *&c.* And it were requir'd to find the Time when all thoſe Sums may be truly diſcharged at one Payment without Loſs, *&c.*

CHAP.

CHAP. XII.

Of Compound Intereſt, and Annuities, &c.

Compound Intereſt is that which ariſes from any Principal and its Intereſt put together, as the Intereſt ſtill becomes due; ſo that at every Payment, or at the Time when the Payments became due, there is created a new Principal; And for that Reaſon it is called Intereſt upon Intereſt, or Compound Intereſt.

As for Inſtance; Suppoſe 100 *l.* were lent out for two Years, at *6 per Cent. per Annum*, Compound Intereſt: Then, at the End of the firſt Year, it will only amount to 106 *l.* As in Simple Intereſt. But for the ſecond Year this 106 *l.* becomes Principal, which will amount to 112 *l.* 7 *s.* $2\frac{1}{2}$ *d.* at the ſecond Year's End, whereas by Simple Intereſt it would have amounted to but 112 *l.*

And altho' it be not lawful to let out Money at Compound Intereſt; yet in purchaſing of Annuities or Penſions, *&c.* And taking Leaſes in Reverſion, it is very uſual to allow Compound Intereſt to the Purchaſer for his ready Money; and therefore it is very requiſite to underſtand it.

Section 1. *Of* Compound *Intereſt.*

Let
- $P =$ the Principal put to Intereſt. } As before.
- $t =$ the Time of its Continuance. } As before.
- $A =$ the Amount of the Principal and Intereſt. } As before.
- $R =$ the Amount of 1 *l.* and its Intereſt for 1 Year, at any given Rate, which may be thus found.

Viz. 100 : 106 : : 1 : 1,06 = the Amount of 1 *l.* at *6 per Cent.*
Or 100 : 107 : : 1 : 1,07 = the Amount of 1 *l.* at 7 *per Cent.*
and ſo on for any other aſſigned Rate of Intereſt.

Then if $R =$ the Amount of 1 *l.* for one Year, at any Rate.
$RR =$ the Amount of 1 *l.* for two Years.
$RRR =$ the Amount of 1 *l.* for three Years.
$R^4 =$ the Amount of 1 *l.* for four Years.
$R^5 =$ the Amount of 1 *l.* for five Years. Here $t = 5$

For $1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5$: &c. in ∺.

That is { As one Pound : is to the Amount of one Pound at one Year's End : : So is that Amount : To the Amount of one Pound at two Year's End, *&c.*

Whence

Whence it is plain, that Compound Intereſt is grounded upon a Series of Terms increaſing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years) does aſſign the Index of the laſt and higheſt Term.
Viz. the Power of R, which is Rt.

Again, As $1 : Rt :: P : PRt = A$ the Amount of P for the Time, that $Rt =$ the Amount of 1 *l.*

That is $\left\{\begin{array}{l}\textit{As one Pound : is to the Amount of one Pound} \\ \textit{for any given Time :: So is any propoſed Prin-} \\ \textit{cipal (or Sum): To its Amount for the ſame Time.}\end{array}\right.$

From the Premiſes, (I preſume) the Reaſon of the following *Theorems*, may be very eaſily underſtood.

Theorem 1. $PRt = A$ As above.

Frome hence the two following *Theorems* are eaſily deduced.

Theorem 2. $\left\{\frac{A}{Rt} = P\right.$. *Theorem* 3. $\left\{\frac{A}{P} = Rt\right.$.

By theſe three *Theorems*, all Queſtions about Compound Intereſt, may be truly reſolved by the Pen only, *viz.* without Tables; Tho' not ſo readily as by the Help of Tables calculated on Purpoſe, As will appear farther on.

Queſtion 1. *What will* 256 l. 10 s. *Amount to in ſeven Years, at* 6 per Cent. per Annum. *Compound Intereſt?*

Here is given $P = 256{,}5$. $t = 7$. and $R = 1{,}06$ which being involved until its Index $= t$ (*viz.* 7) will become $R^7 = 1{,}50363$
Then $1{,}50363 \times 256{,}5 = 385{,}6811 = A = 385$ *l.* 13 *s.* $7\frac{1}{2}$*d*, which is the Anſwer required.

Queſtion 2. *What Principal or Sum of Money muſt be put (or let) out to raiſe a Stock of* 385 l. 13 s. $7\frac{1}{2}$d. *in ſeven Years, at* 6 per Cent. per Annum *Compound Intereſt?*

Here is given $A = 385{,}6811$ $R = 1{,}06$ and $t = 7$ To find P. by *Theorem* 2.
Thus $Rt = 1{,}50363)$ $385{,}6811 = A$ $(256{,}5 = P$.
That is, $P = 256$ *l.* 10 *s.* which is the Principal or Sum as was required.

Queſtion

Question 3. *In what Time will* 256 l. 10 s. *raise a Stock of* (*or amount to*) 385 l. 13 s. 7½ d. *allowing* 6 per Cent. per Annum *Compound Interest?*

Here is given $P = 256,5$ $A = 385,6811$ $R = 1,06$ To find t by the third *Theorem* $\Big\{ Rt = \frac{A}{P}\ \frac{385,6811}{256,5} = 1,50363,$

which being continually divided by $R = 1,06$ until nothing remain; the Number of those Divisions will be $= 7 = t$.

Thus $1,06)\ 1,50363\ (1,41852$ And $1,06)\ 1,41852\ (1,338225$ Again $1,06)\ 1,338225\ (1,262477$. And so on until it become $1,06)\ 1,06\ (1.$ which will be at the seventh Division. Therefore it will be $t = 7$ the Number of Years required by the Question.

Question 4. *If* 256 l. 10 s. *will amount to* (*or raise a Stock of*) 385 l. 13 s. 7½ d. *in seven Years Time; What must the Rate of Interest be,* per Cent. per Annum.

Here is given $P = 256,5,$ $A = 385,6811$ and $t = 7$ Quere R.

By *Theorem* 3. $\Big\{ \frac{A}{P} = Rt = 1,50363.$ As before in the last Question

And if $Rt = R^7 = 1,50365$ Then $R = \sqrt[7]{1,50363}$ which may be thus extracted.

Put		
	1	$r + e = R$ Then
1 Ⓢ 7	2	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$
$2 - r^7$	3	$7r^6e + 21r^5ee = G - r^7$
$3 \div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$
$4 \div$	5	$e = \frac{D}{r + 3e}$ Let $r = 1$ Then $D = 0,0719$

Operation $r = 1,00)\ 0,0719\ (0,06 = e$
$+ 3e = \ ,18 \qquad 708$

Divisor 1,18 (11) to be rejected.

$$\left.\begin{array}{r} \text{First } r = 1,00 \\ + e = 0,06 \end{array}\right\} = 1,06 = R$$

Then $1 : 0,06 :: 100 : 6$ The Rate *per Cent.* required.

The first three Questions may be much more easily perform'd by the following Table, which is only the Amounts of one Pound for Thirty Nine Years.

That

That is, of $R . RR . RRR . R^4 . R^5 .$ and ſo on to R^{39}.

Years $= t$	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Years $= t$	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Years $= t$	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.
1	1,06 $= R$	14	2,2609039557	27	4,8223459407
2	1,1236 $= RR$	15	2,3965581931	28	5,1116866971
3	1,191016 $= R^3$	16	2,5403516847	29	5,4183878990
4	1,26247696	17	2,6927727857	30	5,7434911729
5	1,3382255776	18	2,8543391529	31	6,0881006432
6	1,4185191122	19	3,0255995021	32	6,4533866818
7	1,5036302590	20	3,2071354722	33	6,8405898828
8	1,5938480745	21	3,3995636005	34	7,2510252757
9	1,6894789590	22	3,6035374166	35	7,6860867923
10	1.7908476965	23	3,8197496616	36	8,1472519998
11	1,8982985583	24	4,0489346413	37	8,6360871198
12	2,0121964718	25	4,2918707197	38	9,1542523470
13	2,1329282601	26	4,5493829629	39	9,7035074878

The Title of this Table ſhews its Conſtruction, and its Uſe will eaſily appear by an Example or two.

Example 1. *What will* 375 l. 10s. *amount to in nine Years at* 6 per Cent. per Annum, *&c.*

The Tabular Number againſt *9* Years is 1,689479 which being multiplied with the Principal 375,5 will produce 634,3993 *&c.* *viz.* 634 *l.* 8 *s.* *fere*, being the Amount or Anſwer required.

Example 2. *What Principal (or Sum) muſt be put to Intereſt to raiſe a Stock of* 634 l. 8 s. *in nine Years Time, at* 6 per Cent. per Annum, *&c.*

If the propoſed Stock. (*viz.* 634.4) be divided by the Tabular Number that is againſt the given Number of Years, (*viz.* 9.) the Quotient will be the Principal (or Sum) requir'd. *Viz.* againſt 9 is 1.689479)

Then 1 689479) 634,4 (375,5 = 375 *l.* 10 *s.* the Principal (or Sum) as was required.

Example 3. *In what Time will* 375 l. 10 s. *raiſe a Stock of (or amount to)* 634 l. 8 s. *at* 6 per Cent. *&c.*

Divide

Divide the propoſed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will ſhew the Tabular Number that ſtands over-againſt the Time ſought.

Thus 375,5) 634,4 (1,689479 &c. this Number being ſought in the Table, will be found to ſtand againſt 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; Then take out of that Table the neareſt Number that is leſs, and make it a Diviſor, by which you muſt divide the firſt Quotient; And then ſeek the ſecond Quotient in the Table of Amounts for Days, (which is inſerted a little further on) and it will aſſign the Number of Days. As in this

Example.

In what Time will 563 l. *Amount to* 860 l. *at* 6 per Cent. per Annum, *Compound Intereſt* ?

Anſwer. In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which ſhews the Time to be more (or above) ſeven Years; For over-gainſt 7 Years is 1,50363 which being made the new Diviſor :

Viz. 1,50363) 1,52753 (1,01589, *&c.* this Number is the neareſt Amount to 99 Days.

Note, *If the Stock, Principal, and Time be given; the Rate of Intereſt will be beſt found by extracting the Root,* &c. *As before in the fourth Queſtion.*

The next Thing that I ſhall here propoſe, is to make this Table (which is only calculated for the Rate of 6 *per Cent.*) Univerſally uſeful for all Rates of Compound Intereſt, which I may preſume to ſay, is a new Improvement of my own, being well ſatisfied it never was publiſhed before; and not only ſo, but I have heard ſeveral very good Artiſts affirm it was impoſſible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1{,}06 = R$ the Amount of 1 *l.* for one Year (in the Table) and any other propoſed Amount of 1 *l.* for one Year; which admits of two Caſes.

Caſe 1. If the propoſed Rate be Greater than $1{,}06 = R$, then will $R + x =$ the true Amount of 1 *l.* for one Year at that Rate.

Case 2. But if the proposed Rate be Less than $1,06 = R$, then it will be $R - x =$ the Amount of 1 *l.* &c.

Make $\begin{cases} t-1=b.\ t-2=c.\ t-3=d.\ t-4=f \text{ \&c.} \\ \frac{1}{2}tb=g.\ \frac{1}{3}cg=m.\ \frac{1}{4}dm=n.\ \frac{1}{5}fn=s \text{ \&c.} \end{cases}$

Then will $R^t + tR^bx + gR^cxx + mR^dxxx$ &c. = the Amount of 1 *l.* at the given Rate, for any Time denoted by t. In Case 1.

And $R^t - tR^bx + gR^cxx - mR^dxxx$ &c. = the Amount of 1 *l.* in Case 2.

Which is no more but this, Let $R+x$ Or $R-x$ (which soever it is) be involved (as directed in *Sect.* 5. *Chap.* 2.) to the same Power or Height as the Index t the given Time in the Question denotes: rejecting all the Powers of x above xxx Or $xxxx$ at most, as useless.

Then multiply that Power of $R+x$ Or $R-x$ into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

Example 1. *Suppose it were required to find what* 256 l. *would amount to in fifteen Years, at* 8 l. per Cent. per Annum, *Compound Interest*? Here $t = 15$.

First 100 : 108 :: 1 : 1,08 the Amount of 1 *l.* at 8 *per Cent.*
Next $1,08 - 1,06 = 0,02 = x$. And $R + x = 1,08$ As in Case 1.

Then $R^{15} + 15R^{14}x + 105R^{13}xx + 455R^{12}xxx$ &c. = the Amount of 1 *l.* for 15 Years, at 8 *per Cent.*

Here $x = 0,02$. $xx = 0,0004$. and $xxx = ,000008$
By the Table $R^{15} = 2,396558$

And $\begin{cases} 15R^{14}x = 2,260904 \times 15 \times ,02 & = 0,678271 \\ 105R^{13}xx = 2,132928 \times 105 \times ,0004 & = 0,089583 \\ 455R^{12}xxx = 2,012196 \times 455 \times ,000008 & = 0,007324 \end{cases}$

Sum = 3,171744

Then $3,171744 \times 256 = 811,966464 = A$
That is, 811 *l.* 19 *s.* 4 *d.* *fere* Which is the Answer as was required.

Example 2. *What will* 365 l. *amount to in seven Years, at Four and a half* per Cent. *&c.*

First 100 : 1,045 :: 1 : 1,045 the Amount of 1 *l.* at $4\frac{1}{2}$ *l.* *per Cent.*

Next

Next 1,06 — 1,045 = 0,015 = x. Conſequently $R - x = 1{,}045$ as in Caſe 2.

Then $R^7 - 7R^6x + 21R^5xx - 35R^4xxx$ &c. = the Amount of 1 *l.* for 7 Years, at $4\frac{1}{2}$ *per Cent.*

Here $x = {,}015$. $xx = {,}000225$. and $xxx = {,}000003375$

$$\begin{array}{rl} \text{By the Table } R^7 & = + 1{,}503630 \\ \text{And} \left\{ \begin{array}{l} - 7R^6x \\ + 21R^5xx \\ - 35R^4xxx \end{array} \right. & \begin{array}{l} = - 0{,}148944 \\ = + 0{,}006323 \\ = - 0{,}000141 \end{array} \\ R^7 - 7R^6x + 21R^5xx - 35R^4xxx & = 1{,}360868 \end{array}$$

Then $1{,}360868 \times 365 = 496{,}71682 = A$.

That is 496 *l.* 14 *s.* $3\frac{3}{4}$*d.* is the Anſwer required.

If the Reaſon of theſe two Operations be but well underſtood, it will be very eaſy to conceive how to find P, the Principal, by having A, t, and x given (becauſe R and its Powers are always given by the Table.)

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx \times P = A$ (as above.)

Therefore $\dfrac{A}{R^t \pm tR^bx + gR^cxx \pm mR^dxxx} = P$

Or if A, P, and t be given, x may be found.

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx = \dfrac{A}{P}$ This Æquation being ſolved, (as in *Chap:* 10.) the Value of x will be found; and then either $R + x$, Or $R - x$ will ſhew the Rate of Intereſt, *&c.*

But I ſhall leave the Numerical Operations to the Learner's Practice, ſuppoſing enough done to ſhew how all Queſtions of this kind that are limited by whole Years, may be computed.

And if the Time given or ſought be not terminated by whole Years, but by Weeks, Months, Quarters or Half-Years, &c. for reſolving ſuch Queſtions, the beſt Way will be to reduce thoſe Parts of a Year into Days; that done, find an anſwer according to the Demand of the Queſtion (and agreeing to 1 *l.* as before) for thoſe Number of Days; and in order to that, it will be requiſite to find the Amount of 1 *l.* for one Day, (as in my Compendium of *Algebra*, *Page* 110) which I ſhall here inſert.

Put a = the Amount ſought, then it will be

$1 : a :: a : aa :: aa : aaa :: aaa : aaaa \div$ to a^{365}.

That is { *As one Pound : Is to its Amount for one Day : : So is that Amount : To the Amount of two Days : : And so is that of two Days : To that of three Days. And so on in* $\div$ *to 365 Days.*

Then the last of the Terms will be $a^{365} = 1{,}06$

Put	1	$r + e = a$. And let $r = 1$
1 ⓖ 365	2	$r^{365} + 365 r^{364} e + 66430 r^{363} ee = a^{365} = 1{,}06$
2, in Numb.	3	$1 + 365e + 66430ee = 1{,}06$
3 — 1	4	$365e + 66430ee = 0{,}06$
4 $\div$ 66430	5	$,00549e + ee = 0{,}0000009032 = D$
5 $\div$	6	$e = \dfrac{D}{,00549 + e}$

Operation ,00549) 0,0000009032 (,00016 = e
$+ e =$,00016) 55

1. *Divisor* ,0055 3532 First $r = 1$
2. *Divisor* ,00565 3390 $+ e = 0{,}00016$

$r + e = 1{,}00016$

New $r = 1{,}00016$ for a second Operation. Then

2, in Numb.	7	$1{,}06013401407 + 386{,}887e + 70402{,}172ee = 1{,}06$ Hence it appears that $r - e = a$
Therefore	8	$1{,}06013401407 - 386{,}887e + 70402{,}172ee = 1{,}06$
8 $\pm$	9	$386{,}887e - 70402{,}172ee = 0{,}00013401407$
9 $\div$	10	$,0054953e - ee = ,0000000019035503$
10 $\div$	11	$e = \dfrac{,0000000019035503}{,0054953 - e}$

Operation ,0054953) ,0000000019035503 (,0000003 = e
$- e =$,0000003 164850

Divisor ,0054950 255050 (,0000003464
219800

Last $r = 1{,}00016$ 352503
$- e = 0{,}0000003464$ 329700

$r - e = a = 1{,}0001596536$ 228030
219800

Which being farther pursued to a third Operation, it will be $a = 1{,}00015965358 7453$ &c.

This

This Value of *a* is the Amount of 1 *l.* for one Day from which, if 1 *l.* be ſubtracted, the Remainder =,000159653587 &c. will be the Intereſt of 1 *l.* for one Day. Conſequently, if any propoſed Principal be multiplied into either of theſe, the reſpective Product will be the Amount or Intereſt of that Principal for one Day, at *6 per Cent.* &c.

And that the Amount (or Intereſt) of any Principal or Sum may be eaſily computed for any Number of Days leſs than a Year; I have here inſerted the following Table, which with a great deal of Care (and I believe exactneſs) is calculated from the laſt found (1,000159653587453) Amount of 1 *l.* for one Day. To which alſo is annexed a Table of the Amounts of 1 *l.* for Months.

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
1	1,0001596536	26	1,0041592879	51	1,0081749166
2	1,0003193326	27	1,0043196055	52	1,0083358753
3	1,0004790372	28	1,0044799487	53	1,0084968597
4	1,0006387673	29	1,0046403175	54	1,0086578699
5	1,0007985229	30	1,0048007120	55	1,0088189057
6	1,0009583039	31	1,0049611320	56	1,0089799673
7	1,0011181105	32	1,0051215776	57	1,0091410545
8	1,0012779426	33	1,0052820488	58	1,0093021675
9	1,0014378002	34	1,0054425457	59	1,0094633062
10	1,0015976834	35	1,0056030682	60	1,0096244707
11	1,0017575920	36	1,0057636164	61	1,0097856608
12	1,0019175262	37	1,0059241901	62	1,0099468767
13	1,0020774859	38	1,0060847895	63	1,0101081184
14	1,0022374712	39	1,0062454146	64	1,0102693858
15	1,0023974820	40	1,0064060653	65	1,0104306789
16	1,0025575184	41	1,0065667416	66	1,0105919978
17	1,0027175803	42	1,0067284436	67	1,0107533424
18	1,0028776677	43	1,0068881712	68	1,0109147128
19	1,0030377808	44	1,0070489245	69	1,0110761090
20	1,0031979193	45	1,0072097035	70	1,0112375309
21	1,0033580850	46	1,0073705082	71	1,0113989786
22	1,0035182732	47	1,0075313385	72	1,0115604521
23	1,0036784885	48	1,0076921945	73	1,0117219513
24	1,0038387294	49	1,0078530762	74	1,0118834764
25	1,0039989958	50	1,0080139835	75	1,0120450272

Days

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
76	1,0122066038	116	1,0186908655	156	1,0252166658
77	1,0123682062	117	1,0188535031	157	1,0253803453
78	1,0125298344	118	1,0190161667	158	1,0255440509
79	1,0126914885	119	1,0191788563	159	1,0257077827
80	1,0128531683	120	1,0193415719	160	1,0258715406
81	1,0130148739	121	1,0195043134	161	1,0260353247
82	1,0131766054	122	1,0196670809	162	1,0261991349
83	1.0133383627	123	1,0198298745	163	1,0263629713
84	1,0135001458	124	1,0199926934	164	1,0265268338
85	1,0136619547	125	1,0201555389	165	1,0266907225
86	1,0138237895	126	1,0203184110	166	1,0268546374
87	1,0139856501	127	1,0204813084	167	1,0270185784
88	1,0141475365	128	1,0206442319	168	1,0271825456
89	1,0143094488	129	1,0208071814	169	1,0273465389
90	1,0144713869	130	1,0209701569	170	1,0275105585
91	1,0146333511	131	1,0211331585	171	1,0276746046
92	1,0147953408	132	1,0212961861	172	1,0278386764
93	1,0149573565	133	1,0214592397	173	1,0280027746
94	1,0151193981	134	1,0216223193	174	1,0281668989
95	1,0152814655	135	1.0217854250	175	1,0283310494
96	1,0154435589	136	1,0219485567	176	1,0284952262
97	1,0156056781	137	1,0221117144	177	1,0286594291
98	1,0157678232	138	1,0222748982	178	1,0288236583
99	1,0159299941	139	1,0224381081	179	1,0289879137
100	1.0160921910	140	1,0226013440	180	1,0291521953
101	1,0162544138	141	1.0227646060	181	1,0293165031
102	1.0164166624	142	1,0229278940	182	1,0294808372
103	1,0165789370	143	1,0230912081	183	1,0296451975
104	1,0167412375	144	1,0232545483	184	1,0298095841
105	1,0169035638	145	1,0234179146	185	1,0299739969
106	1,0170659161	146	1,0235813069	186	1,0301384359
107	1,0172282944	147	1,0237447253	187	1,0303029012
108	1,0173906985	148	1.0239081699	188	1,0304673928
109	1.0175513286	149	1.0240716405	189	1,0306319206
110	1.0177155846	150	1.0242351372	190	1,0307964557
111	1,0178780665	151	1,0243986600	191	1,0309610250
112	1,0180405744	152	1.0245622089	192	1,0311256216
113	1,0182031083	153	1,0247257839	193	1,0312902445
114	1,0183656680	154	1,0248893851	194	1,0314548937
115	1,0185282578	155	1,0250530124	195	1,0316195692

Days

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
196	1,0317842709	236	1,0383939484	276	1,0450459680
197	1,0319489990	237	1,0385597318	277	1,0452128133
198	1,0321137534	238	1,0387255415	278	1,0453796853
199	1,0322785341	239	1,0388913778	279	1,0455446584
200	1,0324433410	240	1,0390572405	280	1,0457135092
201	1,0326081742	241	1,0392231298	281	1,0458804611
202	1,0327730339	242	1,0393890454	282	1,0460474397
203	1,0329379198	243	1,0395549876	283	1,0462144449
204	1,0331028321	244	1,0397209563	284	1,0463814768
205	1,0332677706	245	1,0398869515	285	1,0465485353
206	1,0334327355	246	1,0400529732	286	1,0467156206
207	1,0335977268	247	1,0402190214	287	1,0468827325
208	1,0337627444	248	1,0403850961	288	1,0470498711
209	1,0339277883	249	1,0405511973	289	1,0472170363
210	1,0340928586	250	1,0407173250	290	1,0473842283
211	1,0342579552	251	1,0408834793	291	1,0475514469
212	1,0344230782	252	1,0410496601	292	1,0477186923
213	1,0345882275	253	1,0412158674	293	1,0478859643
214	1,0347534033	254	1,0413821012	294	1,0480532631
215	1,0349186054	255	1,0415483616	295	1,0482205885
216	1,0350838338	256	1,0417146485	296	1,0483879407
217	1,0352490887	257	1,0418809620	297	1,0485553196
218	1,0354143699	258	1,0420473021	298	1,0487227252
219	1,0355796775	259	1,0422136687	299	1,0488901576
220	1,0357450115	260	1,0423800618	300	1,0490576166
221	1,0359103719	261	1,0425464815	301	1,0492251025
222	1,0360757587	262	1,0427129278	302	1,0493926150
223	1,0362411719	263	1,0428794007	303	1,0495601543
224	1,0364066116	264	1,0430459001	304	1,0497277204
225	1,0365720776	265	1,0432124261	305	1.0498953132
226	1,0367375701	266	1,0433789787	306	1,0500629327
227	1,0369030889	267	1,0435455579	307	1,0502305790
228	1,0370686342	268	1,0437121637	308	1,0503982521
229	1,0372342059	269	1,0438787961	309	1,0505659519
230	1,0373998041	270	1,0440454551	310	1,0507336786
231	1,0375654287	271	1,0442121407	311	1,0509014320
232	1,0377310798	272	1,0443788529	312	1,0510692121
233	1,0378967573	273	1,0445455918	313	1,0512370191
234	1,0380624612	274	1,0447123572	314	1,0514048529
235	1,0382281916	275	1,0448791493	315	1.0515727134

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
316	1,0517406008	339	1,0556094165	362	1,0594924636
317	1,0519085150	340	1,0557779484	363	1,0596616154
318	1,0520764559	341	1,0559465071	364	1,0598307942
319	1,0522444237	342	1,0561150927	365	1,06
320	1,0524124183	343	1,0562837053		
321	1,0525804397	344	1,0564523448		
322	1,0527484880	345	1,0566210112	Months	The Amounts of 1 *l.* at 6 *per Cent.* For Months.
323	1,0529165631	346	1,0567897045		
324	1,0530846650	347	1,0569584248		
325	1,0532527937	348	1,0571271720		
326	1,0534209493	349	1,0572959594		
327	1,0535891317	350	1,0574647472	1	1,0048675505
328	1,0537573410	351	1,0576335753	2	1,0097587942
329	1,0539255771	352	1,0578024303	3	1,0146738462
330	1,0540938401	353	1,0579713122	4	1,0196128224
331	1,0542621300	354	1,0581402211	5	1,0245758394
332	1,0544304467	355	1,0583091570	6	1,0295630141
333	1,0545987903	356	1,0584781199	7	1,0345744641
334	1,0547671608	357	1,0586471097	8	1,0396103076
335	1,0549355582	358	1,0588161265	9	1,0446706634
336	1,0551039824	359	1,0589851703	10	1,0497556507
337	1,0552724336	360	1,0591542411	11	1,0548653894
338	1,0554409116	361	1,0593233389	12	1,06

The Use of this Table is in all Reſpects like that of whole Years, in finding the Amount of any given Sum for any propoſed Number of Days leſs than a Year.

Example 1. *Suppoſe it were required to find the Amount of* 375*l.* *for* 210 *Days at* 6 per Cent.

The Amount of 1 *l.* for 210 Days is 1,0340928 *&c. per Table.* Then 1,0340928 × 375 = 387,7848 *&c.* = 387 *l.* 15 *s.* 8¼*d.* which is the Amount required.

And the reſt of the Variations may be perform'd juſt as in the Examples of whole Years.

But if the Time given conſiſts of Years, and Parts of a Year, As Quarters, Months, *&c.* Then reduce the odd Time or Parts of the Year into Days; and the Anſwer may then be found at two Operations; As in the following Example.

Example

Example 2. *Suppose it were required to find what* 265 l. *would amount to in five Years and* 135 *Days, at* 6 per Cent. *&c.*

First the Amount of 1 *l.* for $\left\{\begin{array}{l}5 \text{ Years is } 1{,}338225 \text{ \&c.}\\ 135 \text{ Days is } 1{,}021785 \text{ \&c.}\end{array}\right.$

Then 1,338225 × 1,021785 × 265 *l.* = 362,355232 *&c.* being the Amount or Answer required.

Or, if the Amount and Time are given, to find the Principal; then multiply the Amount of 1 *l.* for the Years, and the Amount of 1 *l.* for the odd Days together; And by their Product divide the given Amount, the Quotient will be the Principal required.

Example 3. *What Principal will raise a Stock of* 362 l. 7 s. 1¼d. *Or* 362,355232 l. *in* 5 *Years and* 135 *Days, at* 6 per Cent. *&c.*

The Amount of 1 *l.* for $\left\{\begin{array}{l}5 \text{ Years is } 1{,}338225 \text{ \&c.}\\ 135 \text{ Days is } 1{,}021785 \text{ \&c.}\end{array}\right.$

Then 1,338225 × 1,021785 = 1,367378 *&c.* the Divisor.
Next 1,367378) 362,355232 = *A* (265 *l.* the Principal required,

Again, if the Principal and its Amount are given, to find the Time, at *6 per Cent.* &c. you must divide the Amount by its Principal, and then proceed as in the third *Example, Page* 256, for the Answer required.

But if the Amount and its Principal, with the Time of its being at Interest are given, to find the Rate of Interest; then proceed as in the Fourth Question, *Page* 255 *&c.*

Now in order to make this Table of Amounts for Days, useful for all Rates of Interest (as before in that for Years) you must first find the Simple Interest of 1 *l.* for one Day, both at the given Rate, and also at *6 per Cent.* And call their Difference x.

Thus, Suppose the given Rate were 8 *per Cent. per Annum.* First 100 : 8 :: 1 : 0,08 And 100 : *6* :: 1 : 0,06 the two Simple Interests for one Year.

Then 365) 0,08 (0,00021917 *&c.* the Simple Interest of 1 *l.* for one Day, at 8 *per Cent.*

And 365) 0,06 (0,00016438 *&c.* the Simple Interest of 1 *l.* for one Day, at *6 per Cent.*

Their Difference 0,00005479 = x which may do indifferently well for ordinary small Questions; but where Exactness is required, it will be convenient to make Use of this Proportion.

 Viz.

Viz. $\left\{\begin{array}{l}\text{As the Simple Intereſt of 1 } l. \text{ for one Day, at 6 } per\ Cent.: \\ \text{is to the Tabular Intereſt of 1 } l. \text{ for one Day :: ſo is the} \\ \text{Simple Intereſt of 1 } l. \text{ for one Day, at any given Rate : to} \\ \text{a fourth Number.}\end{array}\right.$

That is, 0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286
Then 0,00021286 — 0,00015965 = 0,00005321 = x.

This x being involved with the reſpective Amounts for Days, in the ſame Manner as was done with thoſe for Years (*vide* Page 258) the Reſult will be the Anſwer to the Queſtion.

Sect. 2. Annuities or Penſions *in Arrear Computed at Compound Intereſt.*

When Annuities, *&c.* are ſaid to be in Arrear, ſee *Page* 248. And I ſhall here make uſe of the ſame Letters to repreſent the ſame things as before in that Page, ſave only that R is here equal to the Amount of 1 *l.* as in *Section* 1. of this *Chap.*

Suppoſe u = the firſt Year's Rent of any Annuity without Intereſt.

Then will $Ru + u = \left\{\begin{array}{l}\text{the Amount of the firſt Year's Rent, and its} \\ \text{Intereſt; More the 2d Year's Rent.}\end{array}\right.$

And $RRu + Ru + u = \left\{\begin{array}{l}\text{the Amount of the 1ſt and 2d Years Rents,} \\ \text{with their Intereſt; More the 3d Year's} \\ \text{Rent, \&c.}\end{array}\right.$

Here $RRu + Ru + u = A$ the Amount of any yearly Rent or Annuity, being forborn three Years. And from hence may be deduced theſe Proportions.

Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$ and ſo on in $\div$ for any Number of Terms or Years denoted by t, wherein the laſt Term will always be uR^{t-1}

Conſequently $A - uR^{t-1}$ = the Sum of all the Antecedents.

And $A - u$ = the Sum of all the Conſequents in the Series.

And therefore it would be $u : uR :: A - uR^{t-1} : A - u$ *Vide* Page 188.

Ergo $Au - uu = RuA - uuR^t$ which being divided all by u, will become $A - u = RA - uR^t$.

From this laſt Æquation it will be eaſy to raiſe the following *Theorems.*

Theorem 1. $\left\{\dfrac{uR^t - u}{R - 1} = A.\right.$ *Theorem* 2. $\left\{\dfrac{RA - A}{R^t - 1} = u.\right.$

Theorem

Theorem 3. $\left\{\frac{RA+u-A}{u}\right. = R^t$. If this Æquation be continually divided by R, until nothing remain, the Number of those Divisions will be t. *See Page* 255.

4. Again $\left\{\frac{A}{u}R - R^t = \frac{A-u}{u}\right.$ If this Æquation be resolved into Numbers, according to the Method proposed in *Sect.* 3. *Chap.* 10. the Root will shew the Value of R.

Question 1. *If* 30 l. *yearly Rent, or Annuity,* &c. *be forborn* (viz. *remain unpaid*) *nine Years; what will it amount to, at* 6 per Cent. per Annum, *Compound Interest?*

Here is given $u = 30$. $t = 9$ And $R = 1{,}06$ To find A. Per *Theorem* 1.

$R^9 = 1{,}689479$ By the Table of Amounts for Years.
$30 = u$

$R^9 u = 50.684370$
$-u = 30,$

$R - 1 = 0.06$) 20,684370 (344,7395 = 344 *l.* 14 *s.* $9\frac{1}{2}$*d.* = A the Amount required.

Question 2. *What yearly Rent, or Annuity,* &c. *being forborn or unpaid nine Years, will raise a Stock of* 344 l. 14 s. $9\frac{1}{2}$d. = 344,7395 *at* 6 per Cent. *&c.*

Here is given $A = 344{,}7395$ $t = 9$. And $R = 1{,}06$ To find u. Per *Theorem* 2.

$AR = 344{,}7395 \times 1{,}06 = 365{,}42387$
$- A = 344{,}7395$

$R^t - 1 = 1{,}689479 - 1 = 0.689479$) 20.68437 (30 = u

Question 3. *In what Time will* 30 l. *yearly Rent raise a Stock or Amount to* 344 l. 14 s. $9\frac{1}{2}$d. *allowing* 6 per Cent. *for the Forbearance of the Payments.*

Here is given $u = 30$ $A = 344{,}7395$ And $R = 1{,}06$ to find t. Per *Theorem* 3.

First $AR + u - A = 365{,}2387 + 30 - 344{,}7395 = 50.68437$
And $u = 30$) 50,68437 (1.689479 = R^t Then
$R = 1{,}06$) 1,689479 (1,593848 And 1.06) 1.593848 (1.50363 and so on until it become 1,06) 1,06 (1. which will be at the ninth Division; therefore $t = 9$.

Or $R^t = 1{,}689479$ being ſought in the Table of Amounts for Years, will be found to ſtand over-againſt 9 Years, which is the Time required.

Queſtion 4. *If* 30 l. per Annum, *being unpaid nine Years, will amount to* 344 l. 14 s. 9½d. *allowing Compound Intereſt for every Payment as it becomes due, what muſt the Rate of Intereſt be* per Cent. *&c.*

Here is given $u = 30$ $A = 344{,}7395$ And $t = 9$ To find R by the laſt of the four Æquations, *Viz.* $\left\{\frac{A}{u}R - R^t = \frac{A-u}{u}\right.$

First $\frac{A}{u} = \frac{344{,}7395}{30} = 11{,}491317$ And $\frac{A-u}{u} = 10{,}491317$

Hence there is the Æquation $11{,}491317\,R - R^9 = 10{,}491317$

Let	1	$r + e = R$ And ſuppoſe $r = 1$
1 ⓔ 9	2	$r^9 + 9r^8e + 36r^7ee = R^9$
1, in Numb.	3	$11{,}491317 + 11{,}491317e = 11{,}491317R$
2, in Numb.	4	$1{,}000000 + 9{,}000000e + 36ee = R^9$
3 — 4	5	$10{,}491317 + 2{,}491317e - 36ee = 10{,}491317$
6 +	6	$36ee = 2{,}491317e$
6 ÷ 36e	7	$e = 0{,}06$ &c.

$\left.\begin{array}{l}\text{First } r = 1\\ + e = 0{,}06\end{array}\right\} = 1{,}06 = R$ $\left\{\begin{array}{l}\textit{As may be eaſily try'd, by involving}\\ \textit{it and ordering it, as the Æquation}\\ \textit{above directs.}\end{array}\right.$

Sect. 3. *To find the* preſent Worth *of Annuities, Penſions, or Leaſes, &c. at Compound Intereſt.*

Let P = the preſent Worth of any Annuity, or Leaſe, *&c.* and the Reſt of the Letters as before.

Then from what has been ſaid in *Section* 3. *Chap.* 11. about purchaſing of Annuities, *&c.* at Simple Intereſt, it will be eaſy to form the like *Theorems* here at Compound Intereſt, *viz.* by combining *Theorem* 1. *Page* 266. And *Theorem* 1. *Page* 254. into one *Theorem.*

For $\left\{\frac{uR^t - u}{R-1} = A\right.$ $\left\{\begin{array}{l}\text{The Amount of any yearly Rent being un-}\\ \text{paid any Number of Years. } \textit{Per Theorem } 1. \text{ of}\\ \text{the laſt Section.}\end{array}\right.$

And $PR^t = A$ $\left\{\begin{array}{l}\text{The Amount of any Principal or Sum being put}\\ \text{to Intereſt, for the ſame Number of Years. } \textit{Per}\\ \textit{Theorem } 1. \textit{ Page } 254.\end{array}\right.$

Hence

Hence it follows, That $PR^t = \frac{uR^t - u}{R - 1}$.

Viz. $PR^{t+1} - PR^t = uR^t - u$ being the very ſame Æquation with that in my Compendium of *Algebra*, Page 112, which is there raiſed from the Conſideration of purchaſing Annuities, or taking of Leaſes, *&c.* to be grounded upon a Rank or Series of Geometrical Proportionals continually decreaſing. Thus $\frac{u}{R}$ is the firſt and greateſt Term; R the common Ratio of all the Terms; and P is the Sum of all the Series.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$ &c. in $\div\!\!\div$ until the laſt Term $= \frac{u}{R^t}$ Then will $P - \frac{u}{R^t}$ be the Sum of all the Antecedents. And $P - \frac{u}{R}$ will be the Sum of all the Conſequents. Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$. Or (in the ſame Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$

which produces $PR^{t+1} - uR^t = PR^t - u$. As above.

From this Æquation may be deduced theſe following *Theorems.*

Theorem 1. $\frac{u - \frac{u}{R^t}}{R - 1} = P$. *Theorem* 2. $\frac{\overline{PK^t \times K :} - PR^t}{R^t - 1} = u$

Theorem 3. $\frac{u}{P + u - PR} = R^t$ Which being continually divided by R, will give t.

Theorem 4. $\frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1}$. The reſolving of this Æquation, will diſcover the Value of R.

Queſtion 1. *What is* 30 *l. yearly Rent, to continue ſeven Years, worth in ready Money, allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $u = 30$. $t = 7$. And $R = 1{,}06$ To find P.

Per *Theorem* 1. *Viz.* $\frac{u}{R^t} = \frac{30}{1{,}50363} = 19{,}9517$.

And $30 - 19{,}9517 = 10{,}0483 = u - \frac{u}{R^t}$

Then

Then $R - 1 = 0{,}06$) $10{,}0483$ ($167{,}4716 = P = 167$ *l.* 9 *s.* 5 *d.* being the Anſwer required.

Queſtion 2. *What Annuity or yearly Rent, to continue ſeven Years, may be purchaſed for* 167 l. 9s. 5 d. *allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

In this Queſtion there is given $P = 167{,}4716$. $t = 7$ And $R = 1{,}06$ To find u. By the ſecond *Theorem.*

First, $PR^t \times R = 251{,}8153 \times 1{,}06 = 266{,}9242$
And $-PR^t = 167{,}4716 \times 1{,}50363 = 251{,}8153$

Then $R^t - 1 = 0{,}50363$) $15{,}1089$ ($30 = u$
That is $u = 30$ *l.* the Anſwer required.

Queſtion 3. *How long may one have a Leaſe of* 30 *l. yearly Rent, for* 167 *l.* 9 *s.* 5 *d. allowing* 6 per Cent *Compound Intereſt to the Purchaſer?*

Here is given $P = 167{,}4716$. $u = 30$. And $R = 1{,}06$ To find t. By the third *Theorem.*

First $P + u = 167{,}4716 + 30 = 197{,}4716$
And $-PR = 177{,}5199$

Then $19{,}9515$) $30 = u$ ($1{,}50363 = R^t$

If this $1{,}50363 = R^t$ be either continually divided by $1{,}06 = R$ until nothing remain (as before in Page 255.) Or if it be ſought in the Table of Amounts for Years, &c: it will diſcover $t = 7$ which is the true Anſwer required.

Queſtion 4. *Suppoſe one ſhould give* 167 *l.* 9 *s.* 5 *d. for the Purchaſe of a Penſion, or Annuity of* 30 *l.* per Annum, *to continue ſeven Years; at what Rate of Intereſt,* per Cent. *would that Purchaſe be made, allowing Compound Intereſt to the Purchaſer?*

In this Queſtion there is given, $P = 167{,}4716$. $u = 30$ And $t = 7$ To find R. Per *Theorem* 4.

The 4th *Theorem* is this Æquation $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^t + 1 \right.$

Which being brought into Numbers, and its Root extracted, as in the 4th Queſtion of the laſt Section; the Value of R will be found 1,06. *viz.* $R = 1{,}06$.

And then it will be, 1 : 0,06 :: 100 : 6 the Rate *per Cent.* as was required.

These

Theſe four Queſtions include all the Varieties that can be propoſed about purchaſing Annuities or Leaſes, &c. which are to be either immediately enter'd upon, or in Poſſeſſion at the Time when the Purchaſe is made.

But ſuch Queſtions as relate to Annuities, or taking of Leaſes, &c. in Reverſion, muſt be parted or divided into two diſtinct Queſtions, each to be ſeparately conſider'd by it ſelf (*See Page* 252.) As in the following Examples.

Example 1. *Suppoſe it were required to compute the preſent Worth of* 75 *l. yearly Rent, which is not to commence or be enter'd upon, until ten Years hence; and then to continue ſeven Years after that Time: at* 6 per Cent. *&c. Compound Intereſt.*

The firſt Work in this Queſtion, is to find what 75 *l. per Annum*, to continue ſeven Years, is worth in ready Money; as if it were to be immediately enter'd upon: And to perform that, there is given $u = 75$. $R = 1{,}06$ And $t = 7$. To find P. As in the firſt Queſtion of this Section.

Thus, $\frac{u}{R^t} = \frac{75}{1{,}503631} = 49{,}8793$ And $75 - 49{,}8793 = 25{,}1207$ $= u - \frac{u}{R^t}$

Then, $R - 1 = 0{,}06)\ 25{,}1207 = 418{,}6789 = 418$ *l.* 13 *s.* $6\frac{3}{4}$*d.* the Anſwer to the firſt Part of the Queſtion.

Then the next Work will be, to find what Principal or Sum being put out ten Years, at 6 *per Cent.* &c. will Amount to 418 *l.* 13 *s.* $6\frac{3}{4}$*d.* Here is given $A = 418{,}6783$ $R = 1{,}06$ And $t = 10$. To find P. *Per Theorem* 2. *Page* 254.

Thus $R^{10} = 1{,}790837)\ 418{,}6783 = A\ (233{,}7884 = 233$ *l.* 15 *s.* 9 *d.* the preſent Worth of 75 *l. per Annum* in Reverſion, &c. As was required.

Example 2. *What Annuity or yearly Rent to be entered upon ten Years hence, and then to continue ſeven Years, may be purchaſed for* 233 *l.* 15s. 9 *d. ready Money, at* 6 per Cent. *&c. Compound Intereſt?*

In the 1ſt Work of this Queſtion there is given, $P = 233{,}7884$. $R = 1{,}06$. And $t = 10$ (the Time which the Annuity is not to be enter'd upon) To find A. *Per Theorem* 1. *Page* 254.

Thus, $PR^t = 233{,}7884 \times 1{,}790847 = 418{,}6783 = A$ the Amount of

of 233 *l.* 15 *s.* 9 *d.* put to Intereſt ten Years, at 6 *per Cent.* &c. Then for the ſecond Work of the Queſtion there is given $P = 418,6783$. $R = 1,06$. And $t = 7$ (the Time that the Annuity is to be enjoy'd) To find u. *Per Theorem* 2. of this *Section.*

Thus $PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$
$\quad -PR^t = 418,6783 \times 1,50363 = 629,5372$

$R^t - 1 = 0,50363)\quad 37,7723\ (75 = u$

That is, $u = 75$ *l.* the yearly Rent required by the Queſtion.

Theſe two Examples of finding P and u do fully ſhew the Method that muſt be uſed in reſolving the two general, and indeed, the moſt uſeful Queſtions about Annuities, or Leaſes in Reverſion: And if there be Occaſion, either the Rate, or the Time, *viz.* R or t, may be found by a due Application of their reſpective *Theorems.*

Note, *That which hath been done in the two laſt Sections about Annuities or yearly Rents,* &c. *at* 6 per Cent. *may alſo be done for any Rate of Intereſt, by applying the Difference of the Rates* (viz. x) *As directed in the firſt Section of this Chapter.*

Now becauſe that Rents and Annuities, *&c.* are uſually paid either by quarterly, or half-yearly Payments, and the Method of computing them by the Pen, may be thought a little Troubleſom; I have inſerted the following Tables of the Amounts of 1 *l.* for each, at 6 *per Cent.*

Half Years $= t$	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Half Years $= t$	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Half Years $= t$	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.
1	1,0295630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1,4185191122	22	1,8982985583
3	1,0913367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430912
10	1,3382255776	20	1,7908476965	30	2,3965581901

Quarterly Amounts.

Quarters of a Year = t.	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Quarters of Years = t.	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Quarters of Years = t.	Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.
1	1,0146738461	21	1,3578024938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708460509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0121964718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6650463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,3382255776	40	1,7908476965	60	2,3965581931

Either of theſe Tables may alſo be made uſeful for any propoſed Rate of Intereſt; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the Difference of the Rate $= x$, &c.

As for Inſtance, Suppoſe any of the aforeſaid Queſtions about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Annum.*

Then $1{,}08 - 1{,}06 = 0{,}02 = x$ for yearly Payments; as before.
Conſequently 2) 0,02 ($0{,}01 = x$ for half Years Payments.
Or 4) 0,02 ($0{,}005 = x$ for Quarterly Payments.

Now theſe Values of x, although they are not really true, yet they may ſerve indifferently well for ſmall Rents; as I have already ſaid, *Page* 265. But if you would work exactly.

Then $\sqrt{1{,}08} = 1{,}0392304845$ &c.
$- \sqrt{1{,}06} = 1{,}0295680141$ *vide Table Page* 272.

Difference $= 0{,}0096624704 = x$ for $\frac{1}{2}$ yearly Payments.

And

And $\sqrt{}:\sqrt{}\,1{,}08 = 1{,}0194263092$ &c.
$-\sqrt{}:\sqrt{}\,1{,}06 = 1{,}0146738461$ See the last Table.

Their Difference $0{,}0047524631 = x$. for Quarterly Payments.

These are the true Values of x. which being involved with their respective Amounts (as before for Years, &c.) according as the Question requires, the Result will be the Answer at 8 *per Cent.* &c. The like may be done for any other Rate, either greater or less than 6.

Now, altho' this Method (See *Page* 257, and 258, &c.) of making the Tables that are only calculated for the Rate of *6 per Cent.* General and Useful for all Rates of Compound Interest, be really true; yet it was rather propos'd, to shew what may be perform'd by the Pen, without a great many Tables of several Rates, than intended for common Practice.

For it must needs be confess'd, that Tables calculated on Purpose for any designed Rate of Interest, are much more ready and useful in common Practice. And therefore since the Legislative Power, have thought fit to reduce the Rate of Interest, and have settled it by an Act of Parliament, at 5 *per Cent.* I've therefore been at the Trouble (which was not a little) to calculate the following Tables for that Rate; but don't think it convenient to take the Tables at *6 per Cent.* out of the Book, because the Examples are all suited to them, and not only so, but they may be found Useful in the taking of Leases for Houses, *&c.* For in those Cases, the Purchaser is always allowed more Interest for his Purchase-Money, than the common Rate paid upon the Loan of Money.

15 Resolved
6799.01.23.45
389
324
650
4929
17223
14022
90024

Ball Rooney

Here

Here follow new *Tables* of the *Amounts* of one Pound, at the *Rate* of 5 *per Cent. per Annum* Compound Interest. For *Years*, *Half Years*, *Quarters*, *Months*, and *Days*.

I. The *Table* of the *Yearly Amounts* of 1 *l.* &c.

Years = t.	The Amounts of 1 *l.* &c.	Years = t.	The Amounts of 1 *l.* &c.	Years = t.	The Amounts of 1 *l.* &c.
1	1,05 = *R*	14	1,97993160	27	3,73345632
2	1,1025 = *RR*	15	2.07892818	28	3.92012914
3	1,157625 = R^3	16	2,18287459	29	4,11613599
4	1,21550625	17	2,29201832	30	4,32194239
5	1,27628156	18	2,40661923	31	4.53803949
6	1.34009564	19	2.52695019	32	4,76494147
7	1,40710042	20	2.65329770	33	5,00318854
8	1,47745544	21	2.78596250	34	5,25334797
9	1,55132822	22	2,92526072	35	5,51601536
10	1,62889463	23	3,07152375	36	5,79181613
11	1 71033936	24	3,22509994	37	6,08140694
12	1.79585633	25	3,38635494	38	6,38547729
13	1,88564914	26	3,55567269	39	6,70475115

II. The *Table* of the *Half Yearly Amounts* of 1 *l.* &c.

Half Yrs. = t.	The Amounts of 1 *l.* &c.	Half Yrs. = t.	The Amounts of 1 *l.* &c.	Half Yrs. = t.	The Amounts of 1 *l.* &c.
1	1,02469507	11	1,30779943	21	1,66912030
2	1,05	12	1 34009564	22	1,71033936
3	1,07592983	13	1 37318940	23	1,75257632
4	1,1025	14	1.40710042	24	1,79585633
5	1.12972632	15	1,44184887	25	1,84020513
6	1,157625	16	1,47745544	26	1,88564914
7	1,18621264	17	1,51394132	27	1.93221539
8	1,22550625	18	1,55132822	28	1,97993160
9	1,24552327	19	1,58963838	29	2,02882616
10	1.27628156	20	1 62889463	30	2 07892818

III. The *Table* of the *Quarterly Amounts* of 1 *l.* &c.

t = Quarters	The Amounts of 1 *l.* &c.	t = Quarters	The Amounts of 1 *l.* &c.	t = Quarters	The Amounts of 1 *l.* &c.
1	1,01227223	21	1,29194439	41	1,64888480
2	1,02469507	22	1,30779943	42	1,66912031
3	1,03727037	23	1,32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020513
11	1,14359059	31	1,45954358	51	1,86278856
12	1,157625	32	1,47745544	52	1,88564914
13	1,17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221539
15	1,20077012	35	1,53252076	55	1,95592799
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036648	57	2,00422978
18	1,24552327	38	1,58963838	58	2,02882616
19	1,26080862	39	1.60914680	59	2,05372439
20	1.27628156	40	1 62889463	60	2.07892818

IV. The *Table* of the *Monthly Amounts* of 1 *l.* &c.

t = Months	The Amounts of 1 *l.* &c.	t = Months	The Amounts of 1 *l.* &c.	t = Months	The Amounts of 1 *l.* &c.
1	1,00407412	5	1.02053728	9	1,03727037
2	1 00816485	6	1,02469507	10	1,04149634
3	1,01227223	7	1,02886981	11	1,04573953
4	1.01639636	8	1,03306155	12	1,05

NOTE : The Amount of one Pound for one Day, is 1,0001336807225 &c. (found as that in *Page* 260.) But in the following Table; I take only nine of those Figures, as being sufficient in Practice, for computing the Interest of any Sum, not exceeding one hundred Millions of Pounds.

V. The

V. The *Table* of the *Dayly Amounts* of 1 *l.* &c.

Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.
1	1,00013368	36	1,00482378	71	1,00953587
2	1,00026738	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066858	40	1,00536119	75	1,01007579
6	1,00080235	41	1,00549558	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643687	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1.00200708	50	1,00670597	85	1.01142688
16	1,00214103	51	1,00684055	86	1,01156209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1,00724438	89	1.01196783
20	1,00267701	55	1.00737903	90	1.01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1 01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1.00468947	70	1,00940093	105	1.01413448

Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.
106	1,01427005	146	1,01970775	186	1,02517459
107	1,01440564	147	1,01984406	187	1,02531164
108	1,01454125	148	1,01998039	188	1,02544870
109	1,01467687	149	1,02011675	189	1,02558578
110	1,01481252	150	1,02025312	190	1,02572288
111	1,01494818	151	1,02038950	191	1,02586000
112	1,01508386	152	1,02052591	192	1,02599714
113	1,01521955	153	1,02066234	193	1,02613430
114	1,01535527	154	1,02079878	194	1,02627147
115	1,01549100	155	1,02093524	195	1,02640866
116	1,01562675	156	1,02107172	196	1,02654588
117	1,01576252	157	1,02120822	197	1,02668310
118	1,01589831	158	1,02134473	198	1,02682015
119	1,01603412	159	1,02148127	199	1,02695762
120	1,01616994	160	1,02161782	200	1,02709490
121	1,01630578	161	1,02175439	201	1,02723221
122	1,01644164	162	1,02189098	202	1,02736953
123	1,01657752	163	1,02202758	203	1,02750686
124	1,01671349	164	1,02216421	204	1,02764422
125	1,01684933	165	1,02230085	205	1,02778160
126	1,01698527	166	1,02243751	206	1,02791899
127	1,01712122	167	1,02257419	207	1,02805640
128	1,01725719	168	1,02271089	208	1,02819384
129	1,01739317	169	1,02284761	209	1,02833129
130	1,01752918	170	1,02298434	210	1,02846875
131	1,01766521	171	1,02312109	211	1,02860624
132	1,01780125	172	1,02325787	212	1,02874375
133	1,01793731	173	1,02339466	213	1,02888127
134	1,01807338	174	1,02353147	214	1,02901881
135	1,01820948	175	1,02366829	215	1,02915637
136	1,01834559	176	1,02380514	216	1,02929395
137	1,01848173	177	1,02394200	217	1,02943154
138	1,01861788	178	1,02407888	218	1,02956916
139	1,01875405	179	1,02421578	219	1,02970679
140	1,01889024	180	1,02435270	220	1,02984445
141	1,01902644	181	1,02448964	221	1,02998212
142	1,01916267	182	1,02462659	222	1,03011980
143	1,01929891	183	1,02476356	223	1,03025751
144	1,01943517	184	1,02490055	224	1,03039524
145	1,01957145	185	1,02503756	225	1,03053298

Days = t	The Amounts of 1 l. &c.	Days = t	The Amounts of 1 l. &c.	Days = t	The Amounts of 1 l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258245
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286622
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03564247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705667
265	1,03605786	305	1,04161236	345	1,04719664

Days

Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.	Days = t	The Amounts of 1 *l.* &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747664	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	*viz.*	1.05

I think it needleſs to ſay any thing of the Uſe of theſe Tables; becauſe I take it for granted, that whoever underſtands the Work of the foregoing Examples, at *6 per Cent.* cannot but know how to make Uſe of theſe Tables of *5 per Cent.* As Occaſion requires.

Thus far concerning ſuch Annuities or Leaſes, *&c.* that are limited by any aſſigned Time; and 'tis only ſuch that can be computed by Theorems or certain Rules. However it may not perhaps be unacceptable, to inſert a brief Account of ſome Eſtimates that have been reaſonably made, by two very ingenious Perſons, about the Proportion, or Difference of Mens Lives, according to their ſeveral Ages; which may be of good Uſe in computing the Values of Annuities, or taking of Leaſes for Lives, *&c.*

Sir *William Petty* in his Diſcourſe made before the Royal Society (*Anno* 1674.) concerning the Uſe of *Duplicate Proportion,* in the Life of Man and its Duration; ſaith, 'That it's found by Experience 'there are more Perſons living of between 16 and 26 Years old, 'than of any other Age or Decade of Years in the whole Life of 'Man (*viz.* 70 or 80 Years.) His Reaſon for that Aſſertion I ſhall omit; but ſuppoſing it true, he thence infers, 'That the Roots of 'every Number of Mens Ages under 16 (whoſe Root is 4) com-'pared with the ſaid Number 4 doth ſhew the Proportion of the 'Likelyhood of ſuch Men reaching the Age of 70 Years.

As for Example, 'Tis 4 Times more likely, that one of 16 Years old ſhould live to 70, than a new-born Babe: 'Tis 3 Times more likely, that one of 9 Years old ſhould attain the Age of 70, than the ſaid Infant. *&c.*

On the other Hand, 'Tis 5 to 4, that one of 25 Years old will die before one of 16: And 6 to 5, that one of 36 will die before one of 25. And ſo on according to the Roots of any other declining Age, compared with (4.6) the Root of 21, which is the Year of Perfection according to the Senſe of our Law, and the Age for whoſe Life a Leaſe is moſt Valuable.

2. The

2. The ingenious and great Mathematician, Captain *Edmund Halley*, (*in Philosoph. Transact. Num.* 196) doth with great Industry and Skill, draw an Estimate of the Proportion of Mens Lives, from the Monthly Tables of the Births and Funerals in *Breslaw*, the Capital City of the Province of *Silesia*; or, as the *Germans* call it, *Schlesia*. Whence he proves, that it is 80 to 1 a Person of 25 Years old will not die in a Year: That it is $5\frac{1}{2}$ to 1, that a Person of 40 will live 7 Years: That a Man of 30 Years old may reasonably expect to live 27 or 28 Years, *&c.*

Now from these and the like Proportions (he justly infers that) the Price of Insurance upon Lives ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For Example; 'tis 100 to 1, that a Man of 20 dies not in a Year, and but 38 to 1, for a Man of 50 Years of Age. And upon these also depends the Valuation of Annuities for Lives: For it is plain, that the Purchaser ought to pay only such a Part of the Value of any Annuity, as he hath Chances that he is living.

And for that Purpose he hath taken the Pains (which was not a little) to compute the following Table, (that shews the Value of Annuities) for every fifth Year of Age to the 70th.

Age	*Year's Purchase.*	*Age*	*Year's Purchase.*	*Age*	*Year's Purchase.*
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same ingenious Gentleman proceeds on, and shews how to estimate or find the Value of two Lives, and then of three Lives, which being too long a Discourse to be recited here, I have, for Brevities Sake, omitted it; and shall only add this serious Observation.

Viz. How unjustly we repine at the Shortness of our Lives, and think our selves wrong'd, if we attain not to old Age; whereas it appears, that the one half of those that are born, die in Seventeen Years Time. For by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616: So that instead of murmuring at what we call a short Life, we ought to account it as a great Blessing that we have survived, per-

haps by many Years, that Period of Life, whereat the one half of the whole Race of Mankind does not arrive.

Sect. 4. *Of Purchasing* Free-hold, *or* Real Estates; *at Compound Interest.*

All Free-hold or Real Estates, are supposed to be purchased or bought to continue for ever (viz. without any limited Time); therefore the Business of computing the true Value of such Estates is grounded upon a Rank or Series of Geometrical Proportionals, continually decreasing, *ad Infinitum.*

Thus, Let P, u, R, denote the same *Data* as in the last *Section.* Then the Series will be, $\frac{u}{R} \cdot \frac{u}{RR} \cdot \frac{u}{R^3} \cdot \frac{u}{R^4} \cdot \frac{u}{R^5}$ and so on in $\div\!\div$ until the last Term $= 0$. Then will $P - 0$ (viz. P) be the Sum of all the Antecedents. And $P - \frac{u}{R}$ will be the Sum of all the Consequents; therefore it will be, $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

This Æquation affords these following *Theorems,*

Theorem 1. $PR - P = u$. *Theorem* 2. $\left\{\frac{u}{R - 1} = P\right.$.

Theorem 3. $\left\{\frac{P + u}{P} = R\right.$.

Example. *Suppose a Free-hold Estate of* 75*l. yearly Rent were to be sold; what is it worth, allowing the Buyer* 6 per Cent. *&c. Compound Interest for his Money?*

In this Question there is given $u = 75$. $R = 1{,}06$ To find P. Per *Theorem* 2. Thus $R - 1 = 0{,}06)\ 75 = u\ (1250\,l. = P$. the Answer required. And so for any of the rest as Occasion requires. But if the Rent is to be paid, either by Quarterly, or Half-yearly Payments;

Then $R = \sqrt{1{,}06}$ for Half-yearly } Payments at 6 *per Cent.*
And $R = \sqrt{:}\sqrt{1{,}06}$ for Quarterly }

Or { $R = 1{,}08$ for Yearly; $R = \sqrt{1{,}08}$ for Half-yearly; $R = \sqrt{:}\sqrt{1{,}08}$ for Quarterly } Payments at 8 *per Cent.*

The like is to be understood for any other proposed Rate of Interest either Greater, or Less than 6 *per Cent.*

The Application of these *Theorems* to Practice, is so very easy, that it's needless to insert more Examples.

A N

AN INTRODUCTION TO THE Mathematicks.

PART III.

CHAP. I.

Of Geometrical Definitions, &c.

Sect. I. Of Lines and Angles.

A *Point* hath no Parts: That is, a Geometrical Point is not any Quantity, but only an Assignable Place in any Quantity, denoted by a *Point*. As at *A*. and *B*. } *A*. *B*.

Such a Place may be conceived so infinitely small, as to be void of Length, Breadth and Thickness; And therefore a Point may be said to have no Parts.

2. A *Line* is called a Quantity of one Dimension, because it may have any supposed Length, but no Breadth nor Thickness, being made or represented to the Eye, by the Motion of a Point.

That is, If the Point at *A*. be moved (upon the same Plane) to the Point at *B*. it will describe a Line, either Right, or Circular, (*viz*. Crooked) according to its Motion.

Therefore the Ends or Limits of a Line are Points.

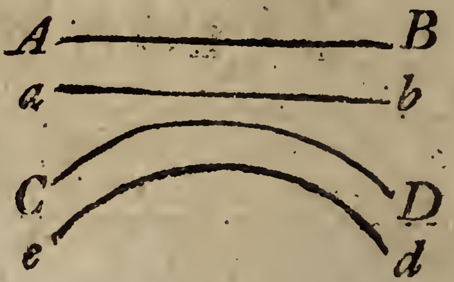
3. A *Right Line*, is that Line which lieth even or strait betwixt those Points that limit its Length being the shortest Line that can be drawn between any two Points. As the Line *A B*. } *A*————*B*

Therefore between any two Points, there can lie, or be drawn but one Right Line.

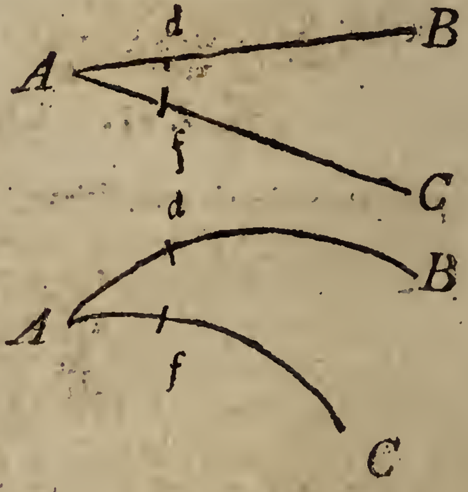
4. A *Circular*, *Crooked* or *Oblique Line*, is that which lies bending between those Points which limit its Length, as the Lines *CD* or *FG*, &c.

Of these Kinds of Lines there are various Sorts; but those of the Circle Parabola, Ellipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account shall be given further on.

5. *Parallel Lines*, as those that lie equally distant from one another in all their Parts, *viz.* such Lines as being infinitely extended (upon the same Plane) will never meet; as the Lines *AB* and *ab*, or *CD* and *cd*.



6. *Lines* not *Parallel*, but *Inclining* (*viz.* leaning) one towards another, whether they are Right Lines, or Circular Lines, will (if they are extended) meet, and make an Angle; the Point where they meet is called the Angular Point, as at *A*. And according as such Lines stand nearer or further off each other, the Angle is said to be less, or greater, whether the Lines that include the Angle be long, or short. That is, the Lines *Ad* and *Af* include the same Angle as *AB*, and *AC* doth; notwithstanding that *AB* is longer than *Ad*, &c.



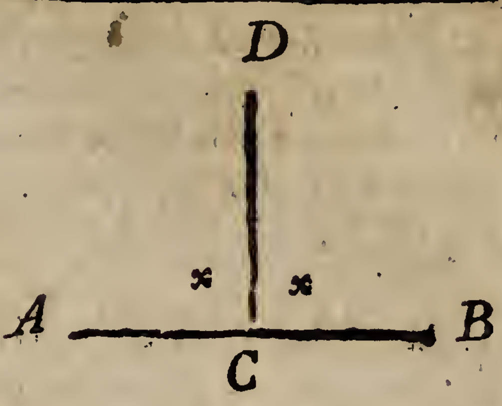
7. All *Angles* included between the Right Lines, are called Right lin'd Angles; and those included between Circular Lines, are called Spherical Angles. But all Angles, whether Right-lin'd or Spherical, fall under one of these three Denominations.

Viz. { A **Right Angle.**
An **Obtuse Angle.**
An **Acute Angle.** }

8. A Right Angle is that which is included betwixt two Lines that meet one another Perpendicular.

That

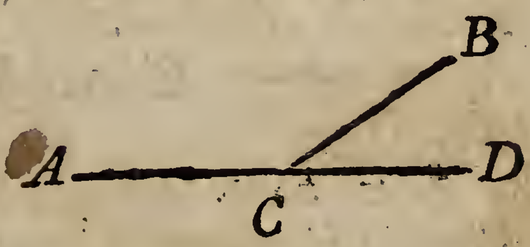
That is, when a Right-line as *D C*, meets with another Right-line as *A B*, so directly as that it neither inclines nor declines to one Side more than the other, but makes the Angles on both Sides of it equal, as at *x*, *x*; Then are those Angles called Right Angles; and the Lines so meeting are said to be Perpendicular to each other.



That is, *A C*, and *C D*, are Perpendicular to *D C*, as well as *D C* is to either or both of them.

9. An *Obtuse Angle* is that which is greater than a Right Angle. Such is the Angle included between the Lines *A C* and *C B*.

10. An *Acute Angle* is that which is less than a Right Angle; as the Angle included between the Lines *C B* and *C D*.



These two Angles are generally called *Oblique Angles*.

Sect. 2. Of a Circle, &c.

Before a Circle *and its* Parts *are defin'd, it will be convenient to give a brief Account of* Superficies *in general.*

1. A *Superficies* or *Surface* is the upper, or very Out-side of any visible Thing. But by Superficies in *Geometry*, is meant only so much of the Out-side of any Thing as is inclosed within a Line, or Lines, according to the Form or Figure of the Thing designed; and it is produced or formed by the Motion of a Line, as a Line is described by the Motion of a Point; thus:

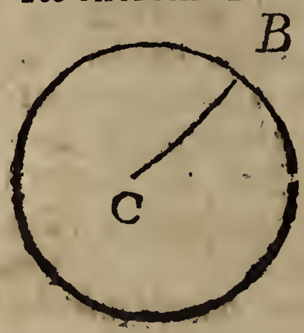
Suppose the Line *A B* were equally moved (upon the same Plane) to *C D*; then will the Points at *A* and *B* describe the two Lines *A C* and *B D*; and by so doing they will form (and inclose the *Superficies* or Figures *A B C D*, being a Quantity of two Dimensions, *viz.* it hath Length and Breadth, but not Thickness. Consequently the Bounds or Limits of a Superficies are Lines.

Note,

Note, *The Superficies of any Figure, is usually called its* **Area**.

2. A *Circle* is a Plain Regular Figure, whose Area is bounded or limited by one continued Line, called the *Circumference* or *Periphery* of the Circle, which may be thus described or drawn.

Suppose a Right Line, as *CB*, to have one of its extreme Points as *C*, so fixt upon any Plane, as that the other Point at *B* may move about it; then if the Point at *B* be moved round about (upon the same Plain) it will be the Circumference or Periphery of that Circle; the Point *C* will be its *Center*, and the contained Space will be it's *Area*, and the Right Line *CB*, by which the Circle is thus described, is called *Radius*.



CONSECTARY.

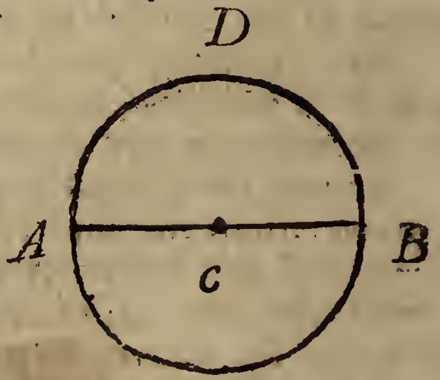
From hence 'tis evident, that an infinite Number of Right Lines may be drawn from the Center of any Circle to touch its Periphery, which will be all equal to one another, because they are all Radius's.

And with a little Consideration it will be easy to conceive that no more than two equal Right Lines can be drawn from any Point within a Circle to touch its Periphery, but from the Center only. (9, e. 3.)

3. *Equal Circles* are those which have equal Radius's; for it's plain by the last Definition that one and the same Radius (as *CB*) must needs describe equal Circles, how many soever they are.

4. The *Diameter* of a Circle, is twice its Radius join'd into one Right Line, as *AB* drawn through the Center *C*, and ending at the Periphery on each Side.

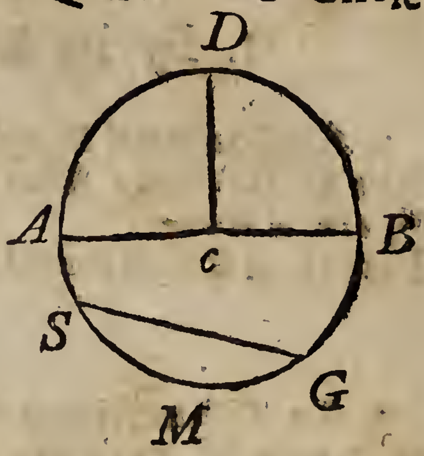
That is, the Diameter divides the Circle into two equal Parts.



5. A *Semicircle* (*viz.* half a Circle) is a Figure included between the Diameter, and half the Periphery cut off by the Diameter; as *ADB*.

6. A

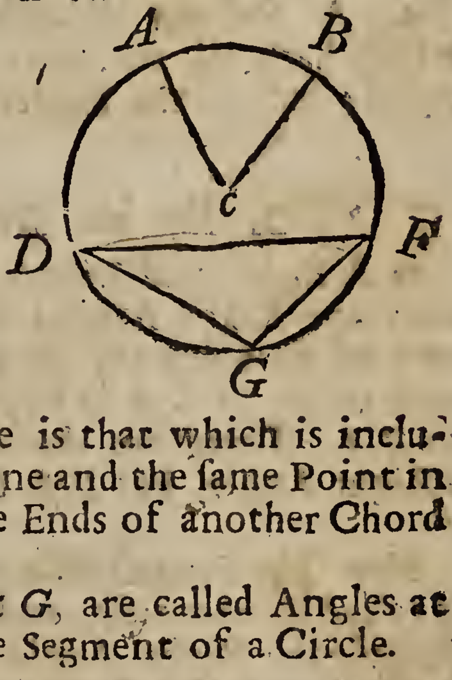
6. A *Quadrant* is half a Semicircle, *viz.* one Quarter of a Circle and it's made by the Radius (as *D C* standing Perpendicular upon the Diameter at the Center *C*, cutting the Periphery of the Semicircle in the Middle, as at *D*. Therefore a Quadrant, or half the Semicircle, is the Measure of a Right Angle.



7. A *Chord Line*, or the Subtense of an Arch, is any Right Line that cuts the Circle into two unequal Parts, as the Line *S G*; and is always less than the Diameter.

8. A *Segment* of a Circle, is a Figure included betwixt the Chord and that Arch of the Periphery which is cut off by the Chord: And it may either be greater, or less than a Semicircle; as the Figure *S M G* or *S D G*.

9. A *Sector* is a Figure included between two Radius's of the Circle, and that Arch of its Periphery where they touch, as the Figure *A C B*, and the Arch *A B* is the Measure of the Angle at *C*, included betwixt the Radius's *A C*, and *B C*.



Note, All Angles of Sectors are called Angles at the Center of a Circle.

10. An *Angle* in the Segment of a Circle is that which is included between two Chords that flow from one and the same Point in the Periphery, as at *D*, and meet with the Ends of another Chord Line, as at *F* and *G*.

That is, the Angles at *D*, at *F*, and at *G*, are called Angles at the Periphery, or Angles standing on the Segment of a Circle.

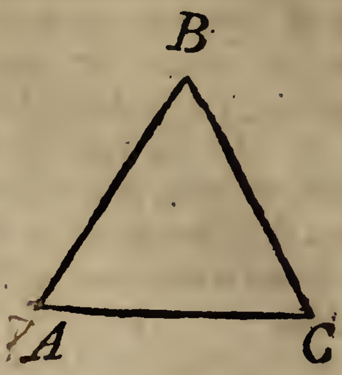
Sect. 3. *Of* Triangles.

There are two Kinds of Triangles, viz. *Plain and Spherical; but I shall not give any Definition of the Spherical, because they more immediately relate to Astronomy.*

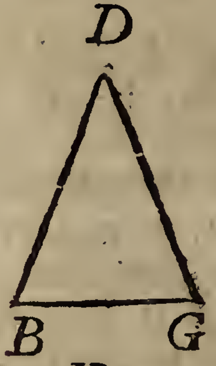
1. A *Plain Triangle* is a Figure whose Area is contained within the Limits of three Right Lines called Sides, including three Angles: And it may be divided, and takes its Name either according to its Sides or Angles.

1. By its Sides.

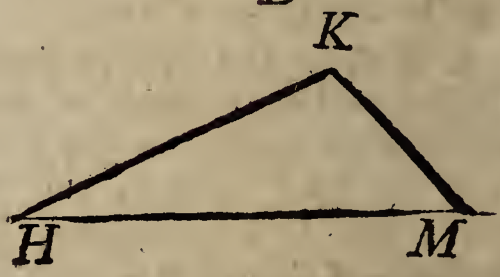
2. An *Equilateral Triangle* is that which hath all its three Sides equal, as the Figure *ABC*
That is, *AB*=*BC*=*AC*.



3. An *Isosceles Triangle* is that which hath only two of its Sides equal as the Figure *BDG*. That is, *BD*=*DG*; but the third Side *BG* may be either greater or leſs, as Occaſion requires.

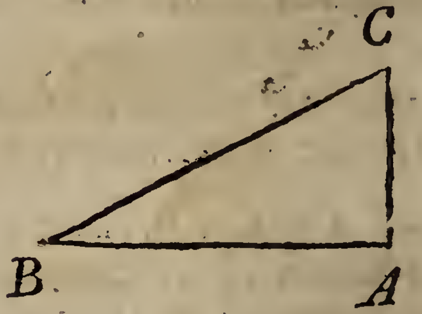


4. A *Scalenous Triangle* is that which hath all its three Sides unequal; ſuch as the Figure *HKM*.



2. By its Angles.

5. A *Right-angled Triangle*, is that which hath one Right Angle; that is, when two of its Sides are Perpendicular to each other, as *CA* is ſuppoſed to be to *BA*. Therefore the Angle at *A*, is a Right Angle, Per *Defi.* 8. *Sect.* 1.



Note, The longeſt Side of every Right-angled Triangle (as *BC*) is called Hypotenuſe, and the longeſt of the other two Sides which include the Right Angle (as *BA*) is called Baſe. The third Side (as *CA*) is called Cathetus or Perpendicular.

6. An *Obtuſe-angled Triangle* is that which hath one of its Angles Obtuſe, and is called an Amblygonium Triangle. Such is the third Triangle *HKM*.

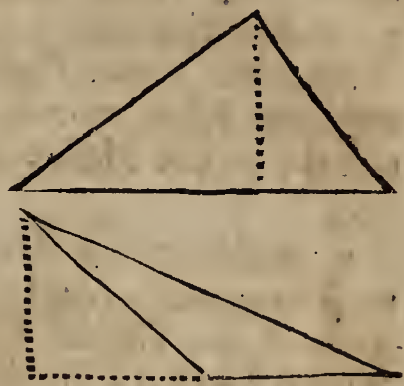
7. An *Acute-angled Triangle* is that which hath all its Angles Acute, and its called an Oxygonium Triangle; Such are the firſt and ſecond Triangles *ABC*, and *BDG*.

Note, All Triangles that have not a Right Angle, whether they are Acute, or Obtuſe, are in general Terms, called oblique Triangles

with

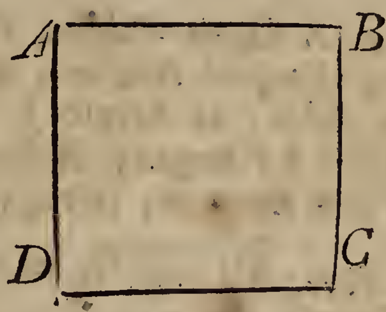
without any other Distinction as before. And the longest Side of every oblique Triangle, is usually called the Base; the other two are only called Sides, or Legs.

8. The *Altitude* or *Height* of any plain Triangle, is the Length of a Right Line let fall perpendicular from any of its Angles, upon the Side opposite to that Angle from whence it falls; and may be either within, or without the Triangle, as Occasion requires, being denoted by the two prick'd Lines in the annexed Triangles.



Sect. 4. Of Four-Sided Figures, &c.

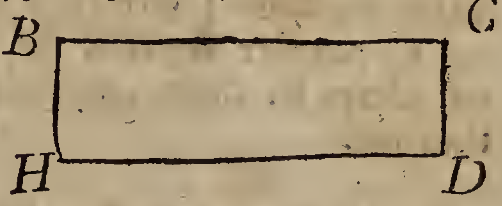
1. A *Square*, is a plain regular Figure, whose Area is limited by four equal Sides all perpendicular one to another.



That is, when $AB = BC = CD = DA$, and the Angles A, B, C, D are all equal, then 'tis usually called a Geometrical Square.

2. A *Rhombus*, or Diamond-like Figure, is that which hath four equal Sides, but no right Angle. That is, a *Rhombus* is a Square moved out of its Right Position, as the annexed Figure.

3. A *Rectangle*, or a right-angled Parallelogram (often called an Oblong, or long Square) is a Figure that hath four right Angles, and its two opposite Sides equal, *viz.* $BC = HD$, and $BH = CD$.

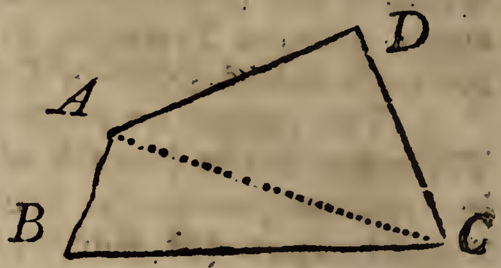


4. A *Rhomboides*, is an oblique-angled Parallelogram, that is, it is a Parallelogram moved out of its right Position, like the annexed Figure.

5. The *Altitude* or Height of any oblique-angled Parallellogram, *viz.* either of Rhombus, or Rhomboides, is a Right-line let fall perpendicular from any Angle upon the Side opposite to that Angle; and it may either be within or without the Figure. As in the prick'd Lines in the annexed Figure.

 6. All

6. All four-sided Figures, which differ from those before-mention'd, are called *Trapezia*.
That is, when they have neither opposite Sides, nor opposite Angles, equal; as the Figure *ABCD*.



7. A Right-line drawn from any Angle in a four-sided Figure, to its opposite Angle, is called a *Diagonal* Line, and will divide the Area of the Figure into two Triangles, being denoted by the prick'd Line *AC* in the last Figure.

8. All Right-lined Figures that have more than four Sides, are called Polygons, whether they be regular, or irregular.

9. A *Regular Polygon*, is that which hath all its Sides equal, standing at equal Angles; and is nam'd according to the Number of its Sides (or Angles). That is, if it have five equal Sides, it is called a *Pentagon*; if six equal Sides, it is called a *Hexagon*; if seven, it is a *Heptagon*; if eight, it is an *Octagon*, &c.

Note, All regular Polygons may be inscribed in a Circle. That is, their angular Points, how many soever they have, will all just touch the Circle's Periphery.

10. An *Irregular Polygon* is that Figure which hath many unequal Sides standing at unequal Angles (like unto the annexed Figure, or otherwise); and of such kind of Polygons there are infinite Varieties; but they may all be reduced to regular Figures, by drawing Diagonal Lines in them, as shall be shewed farther on.

These are the most general and useful Definitions that concern plain or superficial Geometry.
As for those which relate to Solids, I thought it convenient to omit giving any Account of them in this Place, because they would rather puzzle and amuze the Learner than improve him, until he has gained a competent Knowledge in the most useful Theorems concerning Superficies; for then those Definitions may be more easily understood, and will help to form a clearer Idea of their respective Solids, than 'tis possible to conceive of them before; and therefore I have reserved those Definitions until we come to the fifth Part.

Sect.

Sect. 5. *Of such* Terms *as are generally used in Geometry.*

Whatsoever is proposed in Geometry, will either be a *Problem* or a *Theorem*.

Both which *Euclid* includes in the general Term of Proposition.

A *Problem* is that which proposes something to be done, and relates more immediately to Practical, than Speculative Geometry; that is, 'tis generally of such a Nature as to be perform'd by some known or common received Rules, without any Regard had to their Inventions, or Demonstrations.

A *Theorem* is when any common received Rule, or any new Proposition is required to be demonstrated, that so it may from thenceforward become a certain Rule to be relied upon in Practice, when Occasion requires it. And therefore several Rules are often called Theorems, by which Operations in Arithmetick, and Conclusions in Geometry are perform'd.

Note, By *Demonstration* is understood the highest Degree of Proof that human Reason is capable of attaining to, by a Train of Arguments, deduced or drawn from such plain Axioms, and other self-evident Truths, as cannot be denied by any one that considers them.

A *Corollary*, or *Consectary*, is some consequent Truth drawn, or gained from any Demonstration.

A *Lemma* is the Demonstration of some Premises laid down, or proposed as preparative to obviate and shorten the Proof of the Theorem under Consideration.

A *Scholium* is a brief Commentary or Observation made upon some preceding Discourse.

N. B. I advise the young Geometer to be very perfect in the Definitions, *viz.* not to rest satisfied with a bare Remembrance of them, but that he endeavour to gain a clear Idea, or Understanding of the Things defined; and for that Reason, I have been fuller in every Definition than is usual.

And that he may know from whence most of the following Problems, and Theorems contain'd in the two next Chapters are collected; I have all along cited the Proposition, and Book of *Euclid*'s *Elements*, where they may be found.

As for Instance, at Problem 1. there is (3. *e.* 1.) which shews, that it is the third Proposition in *Euclid*'s first Book. The like must be understood in the Theorems.

CHAP. II.

The First Rudiments, or *Leading* and *Preparatory* Problems in *Plain* Geometry.

In order to perform the following Problems, the young Geometer ought to be provided with a thin ſtrait Ruler, made either of Braſs or Boxwood; and two Pair of very good Compaſſes, *viz.* one Pair called three-pointed Compaſſes, being very uſeful for drawing of Figures or Schemes, either with black Lead or Ink, and one Pair of plain Compaſſes with very fine Points, to meaſure and ſet off Diſtances; alſo he ſhould have a very good Steel Drawing-Pen: And then he may proceed to the Work with this Caution; that he ought to make himſelf Maſter of one Problem before he undertake the next, that is, he ought to underſtand the Deſign, and, as far as he can, the Reaſon of every Problem as well as how to do it, and then a little Practice will render them very eaſy, they being all grounded upon theſe following Poſtulates.

Poſtulates or Petitions.

I. That a *Right-line* may be drawn from any one given *Point* to another.

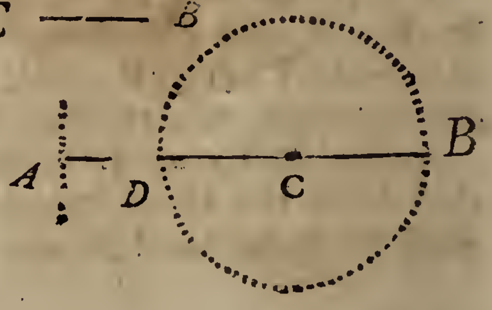
II. That a *Right-line* may be *produced*, *encreaſed*, or made *longer* from either of its Ends

III. That upon any given *Point* (or *Center*) and with any given *Diſtance* (*viz.* with any Radius) a *Circle* may be *deſcribed.*

PROBLEM I.

Two Right-lines being given; To find their Sum and Difference. (3, *e.* 1.)

Let the given Lines be $\left\{ \begin{array}{l} A \text{ ——— } C \\ C \text{ —— } B \end{array} \right.$

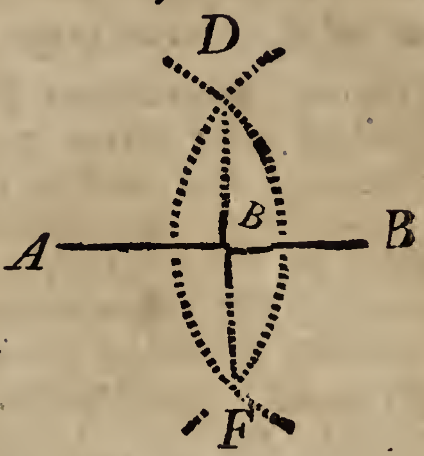


Make the ſhorteſt Line as *CB* Radius and with it deſcribe a Circle; From its Center *C* ſet off the other Line *AC*, and joyn *A C B* with a Right-line. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO-

PROBLEM II.

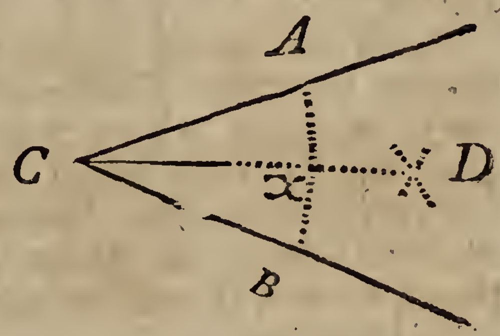
To Bisect, or Divide a Right-line given (as AB) into two equal Parts. (10. *e.* 1.)



From both Ends of the given Line (*viz. A* and *B*) with any Radius greater than Half its Length, describe two Arches, that may cross each other in two Points, as at *D*, and *F*; then joyn those Points *D F*, with a Right-line; and it will Bisect the Line *A B* in the Middle at *C*; *viz.* it will make *A C* = *C B*; as was required.

PROBLEM III.

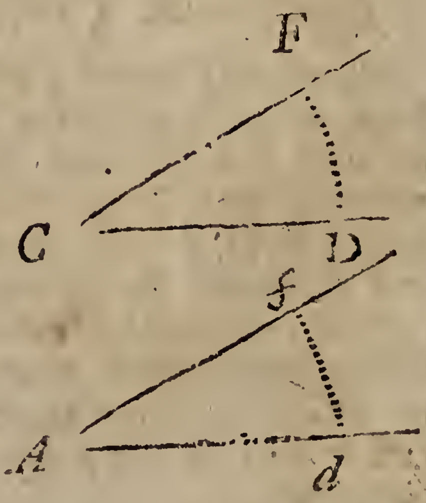
To Bisect a Right-lin'd Angle given, into two equal Angles (9. *e.* 1.)



Upon the Angular Point, as at *C*, describe an Arch as *A B*; and from those Points *A* and *B*, describe two equal Arches crossing each other, as at *D*. Then joyn the Points *C*, and *D*, with a Right line, and it will Bisect the Arch *A B*, and consequently the Angle; as was required.

PROBLEM IV.

At a Point A, in a Right-line given A B. To make a Right-lin'd Angle, equal to a Right-lin'd Angle given C. (23 *e.* 1.)



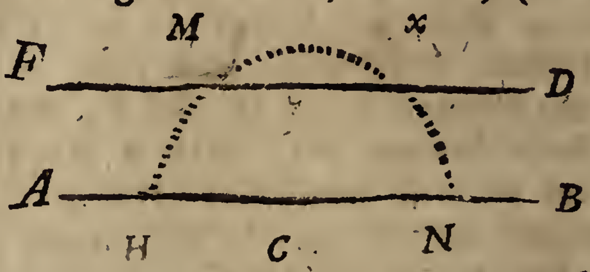
Upon the given Angle Point *C*, describe an Arch, as *F D*, (making *C D* any Radius at Pleasure) and with the same Radius, describe the like Arch upon the given Point *A*, as *f d*. That is, make the Arch *f d* equal to the Arch *F D*; Then joyn the Points *A* and *f*, with a Right-line, and it will form the Angle required.

PRO-

PROBLEM V.

To draw a Right-line, as F D, parallel to a given Right-line A B, that shall pass through any assigned Point, as at x, viz. at any Distance required (31. e. 1.)

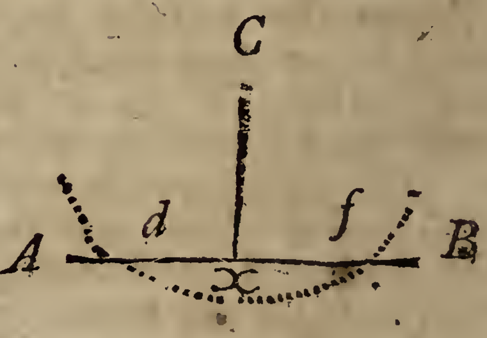
Take any convenient Point in the given Line, as at *C*, (the further off *x* the better); make *C x* Radius, and with it upon the Point *C*, describe a Semi-circle, as *H M x N* Then make the Arch *HM* equal to the Arch *xN*; through the Points *M* and *x*, draw the Right-line *FD*, and it will be parallel to the Line *AC*, as was required.



PROBLEM VI.

To let fall a Perpendicular as C x, upon a given Right-line A B. from any assigned Point that is not in it, as from C. (12. e. 1.)

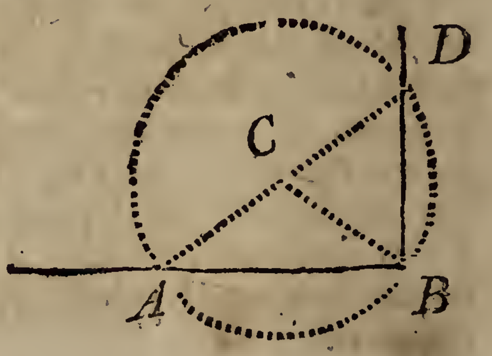
Upon the given Point *C*, describe such an Arch of a Circle, as will cross the given Line *AB* in two Points, as at *d*, and *f*; then Bisect the Distance between those two Points *d*, *f* (*per Prob.* 2.) as at *x*. Draw the Right-line *C x*, and it will be the Perpendicular required.



PROBLEM VII.

To erect or raise a Perpendicular upon the End of any given Right-line, as at B; or upon any o her Point assigned in it. (11. e. 1.)

Upon any Point (taken at an Adventure (out of the given Line, as at *C*, describe such a Circle, as will pass through the Point from whence the Perpendicular must be raised. as at *B* (*viz.* make *C B* Radius); and from the Point where the Circle cuts the given Line, as at *A*. draw the Circle's Diameter *A C D*. Then from the Point *D*. draw the Right-line *D B*, and it will be the Perpendicular as was required.

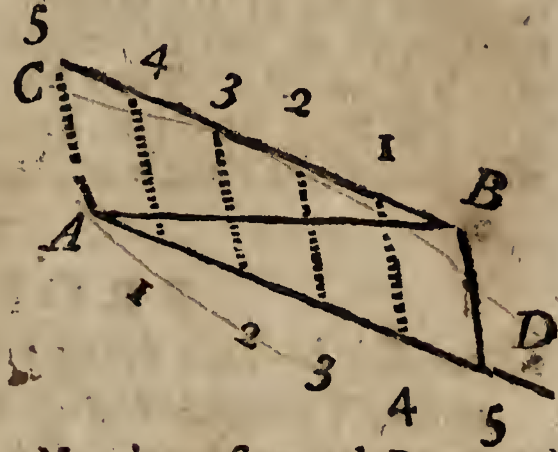


PRO-

PROBLEM VIII.

To divide any given Right-line, as A B, into any proposed Number of equal Parts (10. e. 6.)

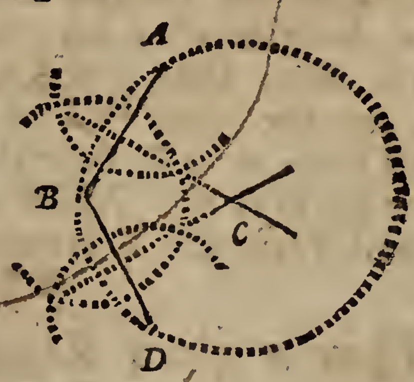
At the extreme Points (or Ends) of the given Line, as at *A* and *B*, make two equal Angles (by *Prob.* 4.) continuing their Sides *A D*, and *B C* to any sufficient Length; then upon those Sides, beginning at the Points *A*, and *B*, set off the proposed Number of equal Parts at pleasure (suppose them 5.) If Right-lines be drawn (cross the given Line) from one Point to the other, as in the annexed Figure, those Lines will divide the given Line *A B* into the Number of equal Parts required.



PROBLEM IX.

To describe a Circle that shall pass (or cut) through any three Points given, not lying in a Right-line.
As the Points A, B, D.

Joyn the Points *B A*, and *B D* with Right-lines, then Bisect both those Lines (*per Problem* 2.) The Point where the bisecting Lines meet, as at *C*, will be the Center of the Circle required.



The Work of this *Problem* being well understood, it will be easy to perform the two following without any Scheme, *viz.* 1. *To find the Center of any Circle given.* (1. e. 3.)

By the last *Prob.* 'tis plain, that if three Points be any where taken in the given Circle's Periphery, as at *A*, *B*, *D*, the Center of that Circle may be found as before.

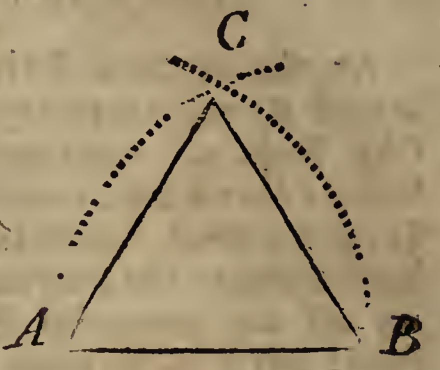
2. *If a Segment of any Circle be given; To complete or describe the whole Circle.* (25. e. 3.)

This may be done by taking any three Points in the given Segment's Arch, and then proceed as before.

PRO-

PROBLEM X.

Upon a Right-line given, as A B; To describe an equilateral Triangle. (1. *e.* 1.)

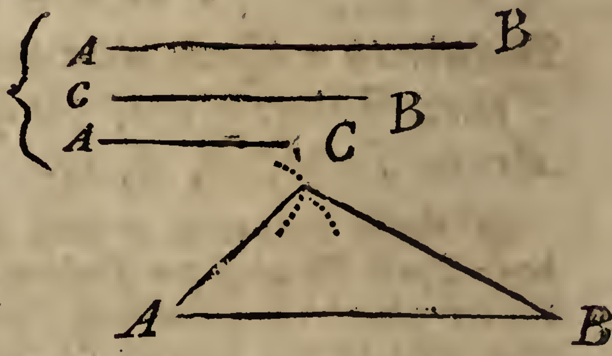


Make the given Line Radius, and with it, upon each of its extreme Points, or Ends, as at *A*, and *B*, deſcribe an Arch, *viz.* *A C*, and *B C*. Then joyn the Points *A C*, and *B C* with Right-lines, and they will make the Triangle required.

PROBLEM XI.

Three Right-lines being given; To form them into a Triangle, (provided any two of them taken together be longer than the third.) (22. *e.* 1.)

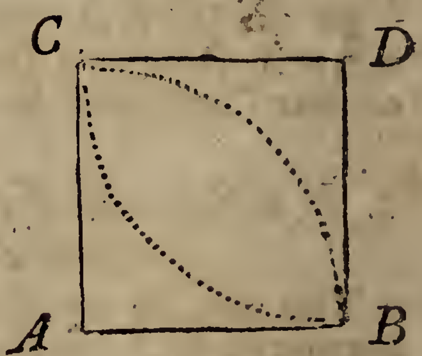
Let the given Lines be



Make either of the ſhorter Lines, as *A C*, *Radius*, and upon either End of the longeſt Line, as at *A*, deſcribe an Arch; then make the other Line *C B* Radius, and upon the other End of the longeſt Side, as at *B*, deſcribe another Arch to croſs the firſt Arch, as at *C*; join the Points *C A*, and *C B* with Right-lines, and they will form the Triangle required.

PROBLEM XII.

Upon a given Right-line, as A, B, To form a Square (46. *e.* 1.)



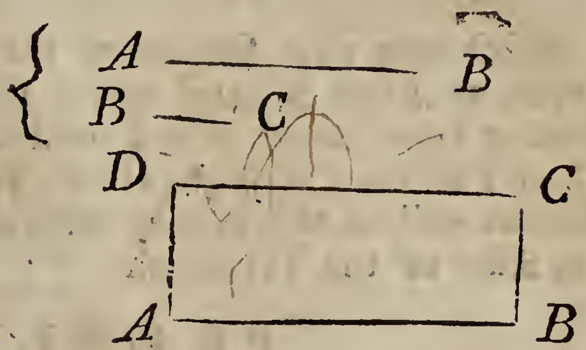
Upon one End of the given Line, as at *B*, erect the Perpendicular *B D*, equal in Length with the given Line, *viz.* make $BD = AB$; that being done, make the given Line Radius, and upon the Points, *A*, and *D*, deſcribe equal Arches to croſs each other, as at *C*; then joyn the Points *C A*, and *C D* with Right-lines, and they will form the Square required.

PRO-

PROBLEM XIII.

Two unequal Right-lines being given; to form or make of them a Right-angled Parallelogram.

Let the given Lines be



Upon one End of the longeſt Line, as at *B*, erect a Perpendicular of the ſame Length with the ſhorteſt Line *B C*; then from the Point *C* draw a Line Parallel, and of the ſame Length to *A B*, *viz.* make *D C* = *A B*. Joyn *D A* with a Right-line, and it will form the Oblong or Parallellogram required.

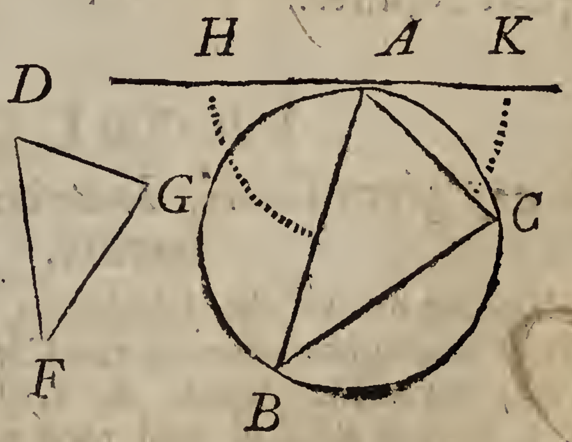
As for Rhombus's, and Rhomboides, to wit, Oblique-angled Parallelograms, they are made, or deſcribed after the ſame Manner with the two laſt Figures; only inſtead of erecting the Perpendiculars, you muſt ſet off their given Angles, and then proceed to draw their Sides parallel, *&c.* As before.

PROBLEM XIV.

In any given Circle, to inſcribe or make a Triangle, whoſe Angles ſhall be equal to the Angles of a given Triangle. As the Triangle F D G, (2. e. 4.)

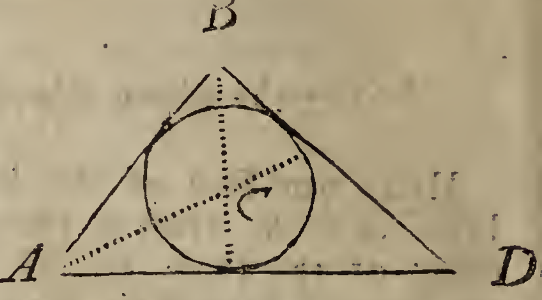
Note, Any Right-lin'd Figure is ſaid to be inſcrib'd in a Circle, when all the Angular Points of that Figure do juſt touch the Circle's Periphery.

Draw any Right-line (as *H K*) ſo as juſt to touch the Circle, as at *A*; then make the Angle *K A C* equal to any one Angle of the given Triangle (as *D F G*); and the Angle *H A B* equal to another Angle of the Triangle (as *D G F*; then will the Angle *B A C* be equal to the Angle *F D G*. Joyn the Points *B* and *C* with a Right-line, and it will form the Triangle required.



PROBLEM XV.

In any given Triangle, as ABD, To describe a Circle that shall touch all its Sides. (4. e. 4.)



Bisect any two Angles of the Triangle, as *A* and *B*, and where the bisecting Lines meet, as at *C*, will be the Center of the Circle required; and its Radius will be the nearest Distance to the Sides of the Triangle.

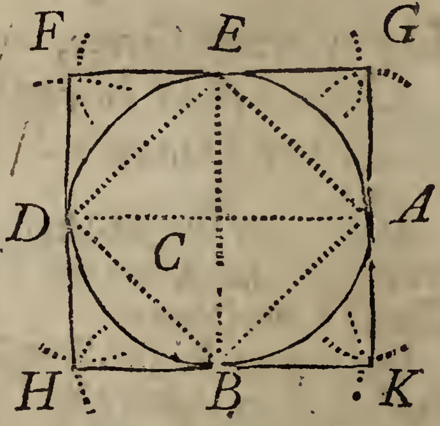
PROBLEM XVI.

To describe a Circle upon any given Triangle. (5. e. 4.)

This Problem is performed in all Respects like the ninth, *viz.* by bisecting any two Sides of the given Triangle; the Point where those bisecting Lines meet, will be the Center of the Circle required.

PROBLEM XVII.

To describe a Square about any given Circle. (7. e. 4.)



Draw two Diameters in the given Circle, as *D A*, and *E B*, at Right-angles, in the Center *C*; and with the Circle's Radius *C A*, describe from the extreme Points of the Diameter, *A*, *B*, *D*, *E*, cross Arches, as at *F*, *G*, *H*, *K*; then joyn those Points where the Arches cross with Right-lines, and they will form the Square required.

PROBLEM XVIII.

In any given Circle, To describe the largest Square it can contain. (6. e. 4.)

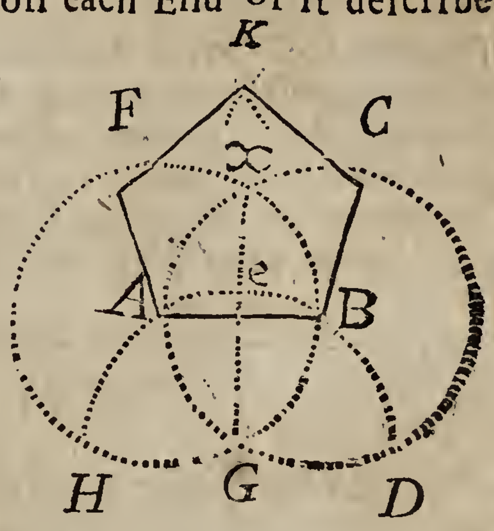
Having drawn the Diameters, as *D A*, and *E B*, bisecting each other at Right-angles in the Center *C*, (as in the last Scheme;) Then joyn the Points *A*, *B*, *D*, and *E*, with Right-lines *viz.* *A B*, *B D*, *D E*, *E A*, and they will be the Sides of the Square required.

PRO-

PROBLEM XIX.

Upon any given Right-line, as A B, To describe a regular Pentagon, or five-sided Polygon.

Make the given Line Radius, and upon each End of it describe a Circle, and through those Points where the Circles cross each other, as at *G*, *x*, draw the Right-line *G e x*, upon the Point *G*, with the same Radius describe the Arch *H A e B D*. Then lay a Ruler upon the Points *D e*, and mark where it crosses the other Circle as at *F*. Again, lay the Ruler upon the Points *H e*, and mark where it crosses the other Circle as at *C*. Then from the Points *F* and *C*, (with the same Radius as before) describe cross Arches, as at *K*, Join the Points *A F*, *F K*, *K C*, and *C B*, with Right-lines, and they will form the Pentagon required.



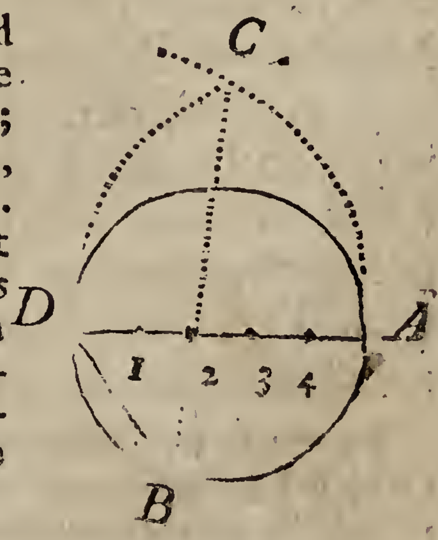
Viz. $AF = FK = KC = CB = AB$. And the Angles at *A*, *B*, *C*, *K*, *F*, will be equal.

PROBLEM XX.

In any given Circle, To describe a regular Pentagon. (11. *e.* 4. *and* 10. *e.* 3.)

Or, in general Terms, to describe any regular Polygon in a Circle.

Draw the Circle's Diameter *D A*, and divide it into so many equal Parts, as the proposed Polygon hath Number of Sides; then make the whole Diameter a Radius, and describe the two Arches *C A*, and *C D*. If a Right-line be drawn from the Point *C*, through the second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's Periphery, as at *B*. Join *D B* with a Right-line. and it will be the true Side of the Pentagon required,



These twenty Problems are sufficient to exercise the young Practitioner, and bring his Hand to the right Management of a Ruler and Compasses, wherein I would advise him to be very ready and exact.

As to the Reason why such Lines must be so drawn, as directed at each Problem; That, I presume, will fully and clearly appear from the following Theorems; and therefore I have (for Brevity's sake) omitted giving any Demonstrations of them in this Chapter, desiring the Learner to be satisfied with the bare Knowledge of doing them only, until he hath fully considered the Contents of the next Chapter; and then I doubt not but all will appear very plain and easy.

CHAP. III.

A Collection of the most useful Theorems *in plane Geometry* Demonstrated.

Note, In order to shorten several of the following Demonstrations, it will be necessary to premise, That

1. The Periphery (or Circumference) of every Circle (whether great or small) is supposed to be divided into 360 equal Parts, called Degrees; and every one of those Degrees are divided into 60 equal Parts, called Minutes, *&c.*

2. All Angles are measured by the Arch of a Circle described upon the Angular Point (*see Defin. 9. Page 287.*) and are esteemed greater or less, according to the Number of Degrees contained in that Arch.

3. A Quadrant or Quarter Part of any Circle, is always 90 Degrees, being the Measure of a Right-angle, (*Defi. 6. Page 287.*) And a Semicircle is = 180 Degrees, being the Measure of two Right-angles.

4. Equal Arches of a Circle, or of equal Circles, measure equal Angles.

To those five general Axioms already laid down in *Page 146.* (which I here suppose the Reader to be very well acquainted with) it will be convenient to understand these following, which begin their Number where the other ended.

Axioms.

Axioms.

6. *Every whole Thing is* **Greater** *than its* **Part.**

That is, the whole Line *AB* is greater than its Part *Ac*, &c.

A——————|——————*B*
c

The same is to be understood of Superficies and Solids.

7. *Every Whole is* **Equal** *to all its* **Parts** *taken together.*

That is, the whole Line *AB* is equal to its Parts *AC* + *cd* + *de* + *eB*.

A——|——|——|——*B*
c *d* *e*

The same is also true in Superficies and Solids.

8. *Those Things which being laid one upon another, do agree, or meet in all their Parts, are equal one to the other.*

But the Converse of this *Axiom*; to wit, that equal Things being laid one upon the other, will meet, is only true in Lines and Angles, but not in Superficies, unless they be alike, *viz.* of the same Figure or Form; as for Instance, a Circle may be equal in Area to a Square; but if they are laid one upon the other, 'tis plain they cannot meet in all their Parts, because they are unlike Figures. Also a Parallelogram, and a Triangle may be equal in their Area's one to another; and both of them may be equal to a Square; but if they are laid one upon the other, they will not meet in all their Parts, *&c.*

Note, *Besides the Characters already explain'd in* Part I. *And in other Places of this Tract, these following are added.*

Viz. $\angle$ denotes an Angle in general, and $\angle\angle$ signifies Angles; $\triangle$ signifies a Triangle; $\square$ signifies a Square; and ▭ denotes a Parallelogram. And when an Angle is denoted by any three Letters, (as *A B C* &c.) the middle Letter (as *B*) always denotes the Angular Point; and the other two Letters (as *AB* and *BC*) denote the Lines, or Sides of a Triangle which include that Angle.

These things being premised, the young Geometer may proceed to the Demonstrations of the following Theorems: Wherein he may perceive an absolute Necessity of being well versed in several things that have been already delivered: And also it will be very advantageous to store up several useful Corollaries, and Lemma's, as they become discovered Truths; for it often happens, that a Proposition cannot be clearly demonstrated, à *priori* or of it self, without a great deal of Trouble. Therefore it will be useful to have Recourse to those Truths that may be assisting to the Demonstration then in Hand.

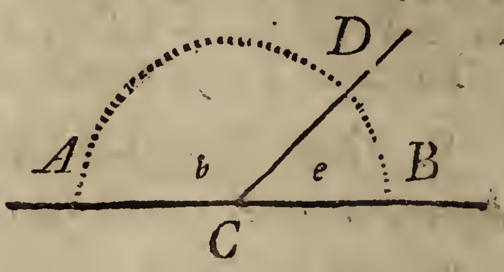
THEO.

THEOREM I.

If a Right-line stand upon (or meet with) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles. (13. e. 1.)

Demonstration.

Suppose the Lines to be *A B* and *D C*, meeting in the Point at *C*; upon *C* describe any Circle at Pleasure. Then will the Arch *A D* be the Measure of the $\angle b$, and the Arch *D B* the Measure of the $\angle e$; but the Arches $AD + DB = 180°$. *viz*. they complete the Semicircle. Consequently, the $\angle b + \angle e = 180°$. Which was to be proved.



Corollaries.

1. Hence it follows, that if the $\angle b = 90°$. then $\angle e = 90°$. but if $\angle b$ be obtuse, then the $\angle e$ will be acute, &c.

From hence it will be easy to conceive, that if several Right-lines stand upon, or meet with any Right-line, at one and the same Point, all the Angles taken together will be $= 180°$. *viz*. Two Right-angles.

THEOREM II.

If two Angles intersect (viz. cut or cross) each other, the two opposite Angles will be equal. (15. e. 1.)

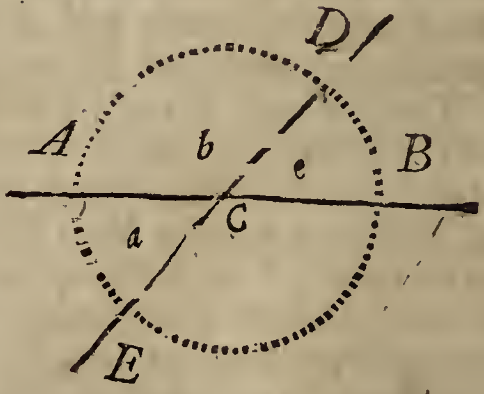
Demonstration.

Let the two Lines be *A B*, and *D E*, Intersecting each other in the Center *C*.

Then $\angle b + \angle e = 180°$ } *per* Last.
And $\angle b + \angle a = 180°$ }

Consequently, $\angle b + \angle e = \angle b + \angle a$, *per* Axiom 5.

Subtract $\angle b$ on both Sides of the Æquation; and it will leave $\angle e = \angle a$.



Again $\angle b + \angle e = 180°$. as before; and $\angle e + \angle C = 180°$. Consequently $\angle e + \angle C = \angle b + \angle e$. Subtract $\angle e$ and then $\angle C = \angle b$. Q. E. D.

Corol-

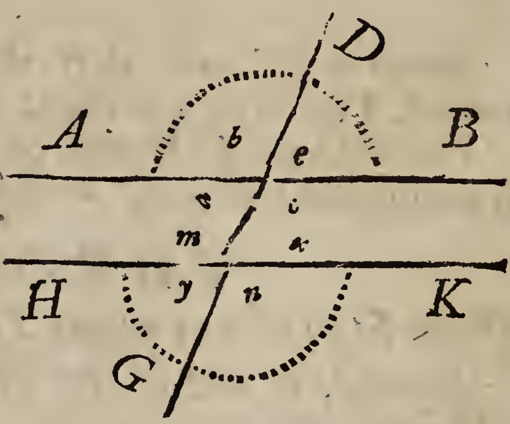
Corollary.

From hence 'tis evident, that if two Lines interſect each other, they will make four Angles, which being taken together, will always be equal to four Right-angles.

THEOREM III.

If a Right-line cut (or croſs) two parallel Lines, it will make the oppoſite Lines equal one to another. (29. *e.* 1.)

Suppoſe the two Lines *A B*, and *H K* to be parallel, and the Right Line *D G*, to cut them both at *C* and *n*. Upon the Point *C* (with any Radius) deſcribe a Semicircle, and with the ſame Radius, upon the Point at *n*, deſcribe another Semicircle oppoſite to the firſt, as in the Figure. Then 'tis plain, and I ſuppoſe very eaſy to conceive, that if the Center *C* were moved along upon the Line *D G*, until it came to the Center at *n*, the two Lines *A B*, and *H K* would meet and concur, *viz.* become one Line (for parallel Lines are as it were but one broad Line.) Conſequently the two Semicircles would alſo meet, and become one entire Circle, like to that in the laſt Demonſtration.



And therefore the $\angle y = \angle x = \angle a = \angle e$ } As before, *per*
And $\angle m = \angle n = \angle b = \angle c$ } laſt *Theorem*

Q. E. D

Corollary.

Hence it follows, that if three, four, or never ſo many parallel Lines are cut or croſs'd by one Right-line, all their oppoſite Angles will be equal.

THEOREM IV.

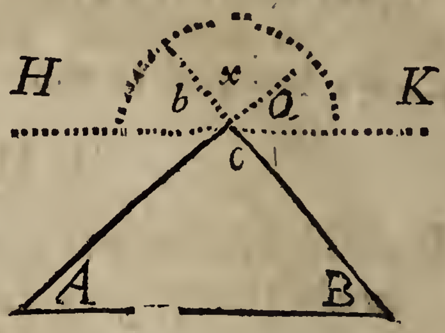
The three Angles of every plane Triangle, are equal to two Right-angles. (32. *e.* 1.)

Conſequently any two Angles of any plane Triangle, muſt needs be leſs than two Right-angles. (17. *e.* 1.)

Demon-

Demonstration.

Let the $\triangle$ *ABC* be proposed; draw the Right-line *HK* parallel to the Side *A B*, just touching the Vertical Angle *C*; and upon the same Angular Point *C*, describe any Semicircle, and produce the Sides *A C*, and *B C* to its Periphery. Then will $\angle b = \angle B$, $\angle a = \angle A$, and $\angle x = \angle C$, *per* last *Theorem*.



But $\angle b + \angle a + \angle x = 180°$, or two Right-angles. Consequently $\angle B + \angle A + \angle C = 180°$. *Per* Axiom 5. Q. E. D.

Corollary.

Hence it follows, that the two Acute Angles of every Right-angled Triangle, are equal to a Right-angle or 90°.

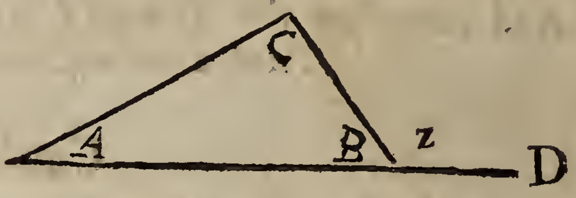
Consequently, if one of the Acute Angles be given, the other is also given; *viz.* 90° — the given $\angle$ leaves the other $\angle$

THEOREM V.

If one Side of any plane Triangle, be continued or produced beyond, or out of the Triangle; the outward Angle will be equal to the two inward opposite Angles. (32. e. 1.)

Demonstration.

Let the Side A B of the $\triangle$ A B C be produced out of the $\triangle$ suppose to D, *&c.* as in the Figure. Then $\angle$ z = $\angle$ A + $\angle$ C, for the $\angle$ B + $\angle$ z = 180°. *Per* Theorem 1. And the $\angle$ B + $\angle$ A + $\angle$ C = 180°. *Per* last Theorem.



Therefore $\angle$ B + $\angle$ z = $\angle$ B + $\angle$ A + $\angle$ C. *Per* Axiom 5. Subtract $\angle$ B on both Sides the Æquation, and it will leave $\angle$ z = $\angle$ A + $\angle$ C. (*Per* Axiom 2.) Q. E. D.

Consequently, the outward Angle (at z *) of any plane Triangle, must needs be greater than either of the inward opposite Angles,* viz. *greater than* $\angle$ A. *or* $\angle$ *c.* (16. e. 1.)

Corollary.

Hence it follows, That if one Angle of any plane Triangle be given, the Sum of the other two Angles is also given, for 180°. —the given $\angle$ = the other two $\angle$ $\angle$.

Theorem

THEOREM VI.

In every plane Triangle, equal Sides subtend (viz. *are opposite to*) *equal Angles.* (5. *e.* 1.)

Consequently, Equal Angles are subtended by Equal Sides. (6. *e.* 1.)

Demonstration.

Suppose the △ *BCD* to be an *Isosceles* △; That is, let *BC* = *CD*. Bisect the ∠ *C*, or (which is all one) make *CA* Perpendicular to *BD*; then will the ∠ ∠ on each Side of it, *viz.* ∠ *BAC* and ∠ *DAC* be Right Angles.

Therefore $\left\{ \begin{array}{l} \frac{1}{2} \angle C + \angle B = 90^\circ \\ \frac{1}{2} \angle C + \angle D = 90^\circ \end{array} \right\}$ Per *Corol.* to *Theorem* 4.

Consequently $\frac{1}{2} \angle C + \angle B = \frac{1}{2} \angle C + \angle D$. Per *Axiom* 5. *Subtract* $\frac{1}{2} \angle C$ from both Sides of the Æquation, and it will leave $\angle B = \angle D$. Per *Axiom* 2. Q. E. D.

COROLLARY.

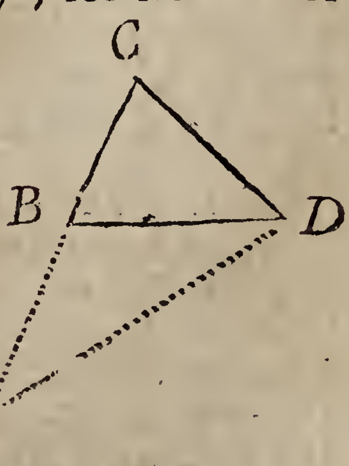
From hence it follows, that the *three Angles of an Æquilateral Triangle*, are equal one to another.

THEOREM VII.

In every plane Triangle, the longest Side subtends the greatest Angle. (18. *e.* 1.)

Consequently, the greatest Angle of any plane Triangle, is subtended by the longest Side.

This *Theorem* is evident by Inspection only; for let one of the Sides of any plane *Triangle*, as *CB*, be produced, suppose to *E*; joyn *DE* with a *Right line*; then 'tis evident, that because *CE* is now made longer than the Side *BC*, therefore the ∠, at *D* is become larger than it was before by the ∠ *BDE*; and it is plain, the longer the Side *CE* had been made, the Angle at *D* would have been the more enlarged.



THEOREM VIII.

If the Sides of two Triangles are equal, the Angles opposite to those equal Sides will be equal. (8. *e.* 1.)

The Truth of this *Theorem* is evident by the two included *Triangles* in the 6th *Theorem*, for they have their respective Sides equal, *viz.* $BC = CD$, $BA = DA$, and CA common to both $\triangle\triangle$. And it is there proved, That the $\angle\angle$ opposite to those equal Sides, are equal, &c. which needs no further *Proof.*

Note, *The Converse of this Theorem holds not true; for the Angles of two Triangles may be equal, and their opposite or subtending Sides unequal, as will appear at Theorem* 12.

COROLLARY.

Hence it follows, that $\triangle\triangle$ *mutually Equilateral*, are also *mutually Equiangular.* And

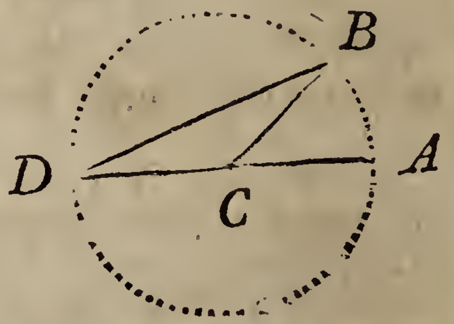
That $\triangle\triangle$ *mutually Equilateral*, are equal one to another. (4, and 26. *e.* 1.)

THEOREM IX.

An Angle at the Center of any Circle, is always double to the Angle at the Periphery, when both the Angles stand upon the same Arch. (20. *e.* 3.) *This Theorem hath three Varieties or Cases.*

Demonstration.

Case 1. Let the Diameter DA and the Line DB, be the two Lines which form the $\angle$ at the *Periphery*; draw the *Radius* BC, then is $\angle BCA$ the $\angle$ at the *Center.* But $\angle BCA = \angle D + \angle B$. Per *Th.* 5. And because $DC = BC$, therefore $\angle D = \angle B$. Per *Theorem* 6. Consequently $\angle BCA = 2\angle D$.



Case 2. Suppose the $\angle BCF$ at the *Center*, to be within the $\angle BDF$ at the *Periphery* (as in the annexed Figure.) Draw the *Diameter* DA.

Then the $\angle BCA = 2\angle BDA$ } *per Case* 1.
And the $\angle FCA = 2\angle FDA$ }

Add these two Æquations together

Then

Then will $\angle BCA + \angle FCA = 2\,\angle BDA + 2\,\angle FDA$ per *Ax.* 1.
But $\angle BDA + \angle FCA = \angle BCF$.
And $2\,\angle BCA + 2\,\angle FDA = 2\,\angle BDF$.
Consequently $\angle BCF = 2\,\angle BDF$.

Case 3. Again, suppose the $\angle BCF$ at the Center to be out of the $\angle BDF$ at the Periphery. From the Angular Point D at the Periphery, draw the Diameter DA.

Then $\left.\begin{array}{l}\angle FCA = 2\,\angle FDA \\ \text{And } \angle BCA = 2\,\angle BDA\end{array}\right\}$ per *Case* 1.
Subtract this last Æquation from the other, and it will leave
$\angle FCA - \angle BCA = 2\,\angle FDA - 2\,\angle BDA$ Per *Axiom* 2.
But $\angle FCA - \angle BCA = \angle FCB$. And $2\,\angle FDA - 2\,\angle BDA = 2\,\angle FDB$. Consequently $\angle FCB = 2\,\angle FDB$. Q. E. D.

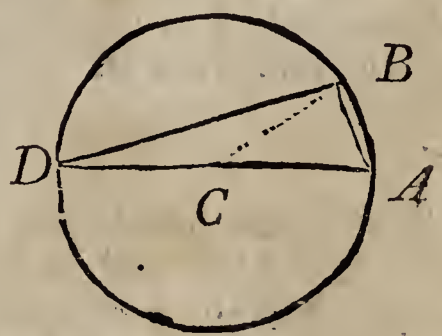
COROLLARY.

Hence it is evident, That all *Angles* at the *Periphery*, which stand on the same *Segment* or *Arch* of a *Circle*, or upon equal *Arches*, are equal one to another. (21. *e.* 3.)

THEOREM X.

An Angle in a Semi-circle, is a Right Angle. (31. *e.* 3.) *That is, if the Diameter of any Circle be the Side of a Triangle, and the Angle opposite to that Side be any where in the Circle's Periphery, it will be a Right Angle.*

Demonstration.



Let DA be the Diameter, and DBA the $\triangle$, then $\angle B = 90°$. Draw the Radius BC, then is the $\angle DBA = \angle D + \angle A$.
For $\angle CBD = \angle D$ and $CBA = \angle A$. Per *Theorem* 6
Therefore $\angle DBA = \angle CBD + \angle CBA$. Per *Axiom* 5.

Again, $\angle DBA + \angle D + \angle A = 180°$. Per *Theorem* 4.
Consequently $\angle DBA = 90°$. or a Right Angle. Q. E. D.

COROLLARIES.

1. Hence it will be easy to conceive, that an *Angle* made in any *Segment* less than a *Semi-circle*, will be *obtuse* or greater than a Right Angle.

2. And an *Angle* made in any *Segment* greater than a *Semi-circle*, must consequently be *acute*.

THEOREM XI.

In any Right-angled Triangle, the Square which is made of the Hypothenuse, or Side subtending the Right Angle, is equal to both the Squares which are made of the Sides including the Right Angle. (47. *e.* 1.)

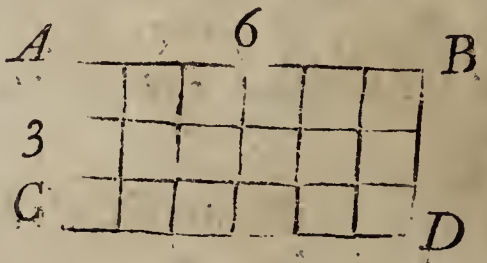
There are several Ways of demonstrating this noble and useful Theorem; but I presume none more easy to be understood by a Learner, than that which I shall here propose; And in order thereto, it will be requisite to premise the following *Lemma's*.

Lemma 1.

A *Right-line* is said to be *multiplied* with a *Right-line* when either a *Square*, or other *Right-angled Parallelogram* is made of the two *Lines*.

That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of those *Numbers* which express the *Measure* of its Sides.

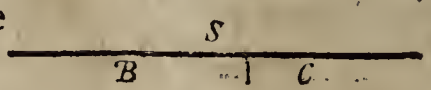
Thus, if $AB = 6$ Inches,
And $AC = 3$ Inches;
Then $AB \times AC = 6 \times 3 = 18$ square Inches, which is the Area of the Parallelogram $ABCD$.



Lemma 2.

If a Right-line be any wise cut into two Parts, the Square of the whole Line will be equal to the Squares of each Part, and a double Rectangle or Parallelogram made of both the Parts. (4. *e.* 2.)

That is, if the Line S, be cut into the two Parts B and C

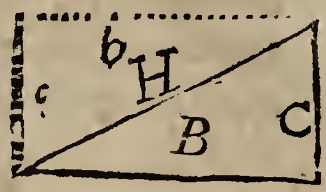


Then is $S = B + C$ but if both the Sides of the Æquation be involved, it will be $SS = BB + 2BC + CC$.

Lemma

Lemma 3.

The Area of every Right-angled Triangle, is half the Parallelogram made of its Base and Perpendicular.



For $BC =$ the Area of the whole Parallelogram, by the first *Lemma*. And $\triangle BCH + \triangle bcH =$ the Parallelogram; But $B = b$ and $C = c$.

Therefore $\frac{1}{2} BC =$ the Area of each $\triangle$ viz. $\frac{1}{2} BC + \frac{1}{2} bc = BC$.

These Things being premised, let us suppose, the Triangle BCH to be a Right angled Triangle, *Viz.* the Side C perpendicular to the Side B. Then will $BB + CC = HH$.

Demonstration

Make a Square whose Side is $= B + C$, and draw the included Square whose Side is $= H$, as in the Scheme. Then will the Area of the great Square be equal to the Area of the four Triangles $+ HH$, but the Area of each $\triangle = \frac{1}{2} BC$ Per *Lemma* 3. Therefore the 4 $\triangle$'s $= \frac{1}{2} BC \times 4 = 2 BC$. Consequently, the Area of the great Square is $HH + 2 BC$. Involve $B + C$, and it will be $BB + 2BC + CC =$ the Area of the great Square. Per *Lemma* 2. Consequently $HH + 2BC = BB + 2BC + CC$. Per *Axiom* 5. Subtract $2BC$ from both Sides of the Æquation, and there will remain $HH = BB + CC$. Q. E. D.

To Illustrate this *Theorem* by *Numbers*; Let us Suppose $C = 3$. $B = 4$. and $H = 5$.
Then will $CC = 9$. $BB = 16$. and $HH = 25$.
Consequently, $BB + CC = HH = 16 + 9 = 25$.

CONSECTARY.

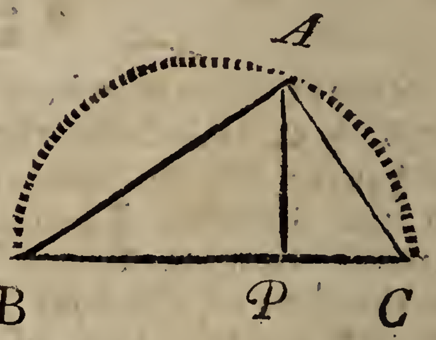
From this admirable *Theorem* (said to be first invented by *Pythagoras*) is deduced the Method of adding, and subtracting Squares, Parallelograms, Circles, *&c.*

THEO-

THEOREM XII.

In any Right-angled Triangle, a Perpendicular being let fall from the Right Angle upon the Hypothenuse, will divide the Triangle into two Right-angled Triangles, which will be both similar or alike to the first Triangle, and to each other. (8. *e.* 6.)

Note, *All plane Triangles are said to be similar,* viz. *Alike when each single Angle in one of the Triangles, is equal to each single Angle of the other; but if any two single Angles of one Triangle are equal to two single Angles of the other; the third Angles will be equal. Per* Theorem 4.

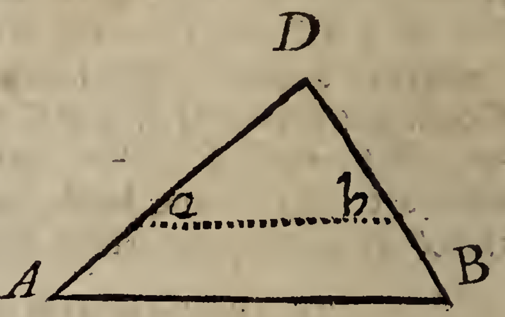


1. In the Right-angled $\triangle$ B*AC*, Let *AP* be supposed Perpendicular to the Hypothenuse B*C*, Then $\angle$ B*AP* $=$ $\angle$ *C*.
For $\angle$ B*AP* $+$ $\angle$ B $= 90°$.
And $\angle$ B $+$ $\angle$ *C* $= 90°$. Per *Corollary* to *Theorem* 4.

Therefore $\angle$ B*AP* $=$ $\angle$ *C*. Per *Axiom* 5.

Again, $\angle$ *PAC* $+$ $\angle$ *C* $= 90°$. and $\angle$ B $+$ $\angle$ *C* $= 90°$. Therefore $\angle$ *PAC* $=$ $\angle$ B, &c. Consequently the $\triangle$ B*AP* is alike to the $\triangle$ *ACP*; And each is alike to the whole $\triangle$ B*AC*.

2. Or if a Right-line be drawn parallel to one of the Sides of any plane Triangle (viz. within it) it will cut off a Triangle similar or alike to the whole Triangle. Thus,



In the $\triangle$ *A*B*D*, draw the Right-line *a b* Parallel to the Side *A*B; Then will the included $\triangle$ *aDb*, be like the $\triangle$ *AD*B. For $\angle$ *a* $=$ $\angle$ *A*, and $\angle$ *b* $=$ $\angle$ B. Per *Theorem* 3. and $\angle$ *D* is common to both the Triangles; *Ergo, &c.*

THEOREM XIII.

If two Triangles are alike, their like Sides will be Proportional.

That is, those Sides which subtend the equal Angles, as also those Sides which are about the equal Angles, will be Proportional

nal to each other; And consequently, if any two Triangles have their Sides Proportional, their Angles are equal. (4,5,6,7.e. 6.)

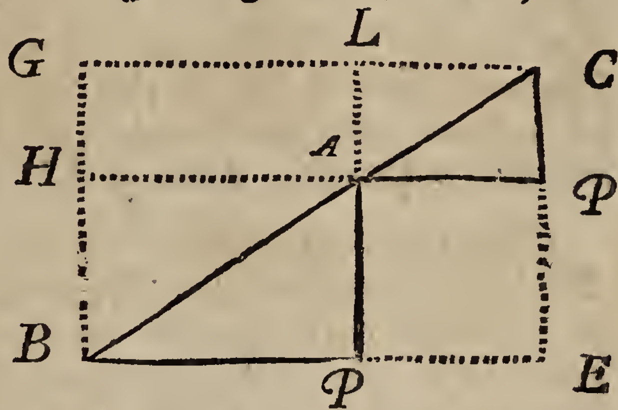
Demonstration.

Let the Similar Triangles in the Scheme of the last *Theorem* be here propos'd again.

Then it will be $BP : AP :: AP : CP$. According to this *Theorem*. *Ergo* $BP \times CP = AP \times AP$.

Demonstration.

Let us suppose the aforesaid Right-angled $\triangle$ BAC, cut through the Perpendicular AP, and there open'd until the Sides BA and CA become one Right-line: Let the Sides BP and CP be continued until they meet in E: Then complete the Parallelograms by drawing the Parallel Lines GLC, HAP, GHB, and LAP, as in the Figure.



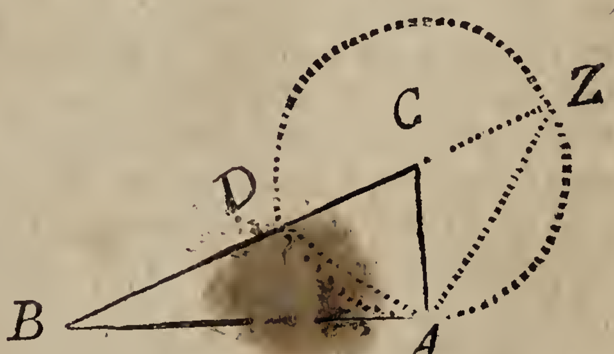
Then it is evident, that the $\triangle$ $BHA = \triangle$ BPA, and the $\triangle$ $CPA = \triangle$ CLA; also that the $\triangle$ $BEC = \triangle$ BGC, because all their respective Sides are equal.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA + \triangle CPA + \square APEP$. Now, if from both Sides of this Æquation there be subtracted the equal $\triangle\,\triangle$, there will remain $\square HGLA = \square APEP$.

But $\square HGLA = BP \times CP$, and $\square$ $APEP = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be prov'd.

Or otherwise, thus:

Suppose the $\triangle$ BAC to be Right-angled at A, upon the $\angle$ Point C with the Radius CA, describe a Circle, and continue the Hypothenuse BC to Z; join ZA and AD with Right Lines; then will the $\triangle$ BAD be like to the $\triangle$ BZA. For $\angle DAB + \angle DAC = 90°$. By Construction.



And $\angle ZAC + \angle DAC = 90°$. By *Theorem* 10.
Therefore $\angle DAB + \angle DAC = \angle ZAC + \angle DAC$. By *Axiom* 5.
Subtract $\angle DAC$ from both Sides of the Æquation and there will remain $\angle DAB = \angle ZAC$. But $\angle ZAC = \angle CZA$.

By

By *Theorem* 6. And $\angle B$ is common to both $\triangle\triangle$.
Therefore $\angle BDA = \angle BAZ$. By *Theorem* 4.
Consequently $\triangle BAD$ is like to $\triangle BZA$.

Let the Sides $\left\{\begin{array}{l} BA = b \\ BC = h \\ CA = c \end{array}\right\}$ $\left\{\begin{array}{l} \text{Then } bb + cc = hh, \text{ Per } \textit{Theor.} \text{ 11.} \\ \text{Consequently } bb = hh - cc. \\ \text{Which gives the following Analogy,} \end{array}\right.$

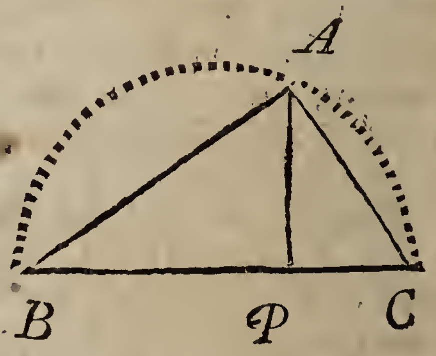
Viz. $b : h + c :: h - c : b$. That is, BA : BZ :: BD : BA.

Q. E. D.

COROLLARIES.

1. Hence it is evident, that in any Right-angled Triangle, a Perpendicular being let fall from the Right Angle upon the Hypothenuse, will be a mean Proportional between the Segments of the Hypothenuse. That is, BP : PA :: PA : PC.

2. The Base (BA) is a mean Proportional between the Hypothenuse (BC) and that Segment of the Hypothenuse next to the Base, (*viz.* BP.) That is, BC : BA :: BA : BP.



3. The Cathetus (AC) is a mean Proportional between the Hypothenuse (BC) and that Segment of the Hypothenuse next to the Cathetus, (*viz.* PC.) That is, BC : AC :: AC : PC.

SCHOLIUM.

I have been larger upon this most excellent *Theorem*, in giving a double Demonstration of it, because it is so universally useful in all Parts of the Mathematicks: For the Business of *Trigonometry* (both Plane and Spherical) wholly depends upon it; and therefore one may truly say, that *Astronomy*, *Dialling*, *Navigation*, *Surveying*, *Optics*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des Cartes* takes particular Notice, as you may find in Dr. *Pell*'s *Algebra*, Page 65, whose Words are these,

" *Des Cartes*, in a Letter not yet printed, writes thus: In
" searching the Solution of Geometrical Questions, I always
" make use of Lines Parallel and Perpendicular, as much as
" is possible, (he means as many Lines as are useful) and I con-
" sider no other *Theorems* but these two, [the Sides of like
" Triangles have like Proportion]. And [in Rectangle Triangles,
" the

" the Square of the greatest Side is equal to the Squares of the two " other Sides.] And I am not afraid to suppose many unknown " Quantities, that I may reduce the propos'd Question to such " Terms, as to depend on no other Theorems but these two.

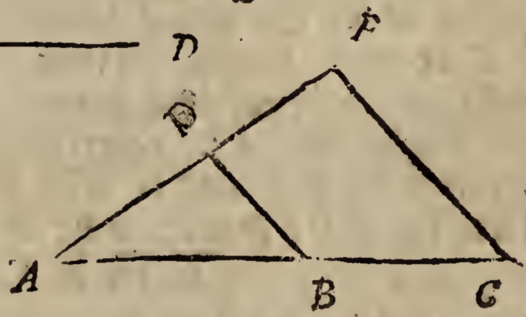
This I thought convenient to insert, that the young Learner may see how the Great *Des Cartes* esteem'd these two Theorems, *viz.* the last, and Theorem 11. For, in Truth, all the preceding Theorems are only (as it were) Preparatives to these two.

This last Theorem demonstrates the Reason of the Method used in finding out proportional Lines, as in the three following Problems.

PROBLEM I.

Two Right-lines being given, to find a third Line in Proportion to them. (11. e. 6.)

Let the two Lines be $\begin{cases} A \text{———} B \\ A \text{———} D \end{cases}$



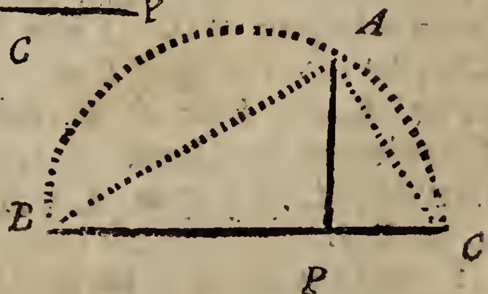
Set the two given Lines at any Angle in the Point *A*, and produce the Line *A B* to *C*, making *B C* = *A D*; join the Points *B D* with a Right-line, and draw *C F* parallel to *B D*; then will the △ *A B D* be like the △ *A C F*.

Therefore *A B* : *B C* (= *A D*) : : *A D* : *D F*, which is the third Proportional requir'd.

THEOREM II.

Two Right-lines being given, to find a Mean proportional Line between them. (13. e. 6.)

Let the given Lines be $\begin{cases} B \text{———} P \\ P \text{———} C \end{cases}$



Join the two given Lines into one, *viz.* make *B C* = *B P* + *P C*, and upon *B C*, as Diameter, describe a Semicircle; then upon the Point *P*, where the 2

Lines meet, erect a Perpendicular to touch the Circle's Periphery, as PA, and it will be the Mean Proportional requir'd.

Viz. $BP : AP :: AP : PC$.

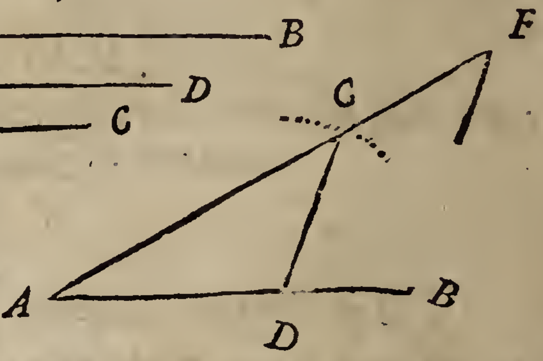
By this Problem it is easy to conceive how to make a Square equal to any given Parallelogram, (14. *e.* 6.)

For if BP be the Length, and PC the Breadth of the given Parallelogram, then will AP be the Side of the Square, equal in Area to that Parallelogram.

PROBLEM III.

Three Right-lines being given, to find a fourth Proportional Line, (12. *e.* 6.)

Suppose the three Lines $\begin{cases} A \text{———} B \\ A \text{———} D \\ D \text{———} C \end{cases}$

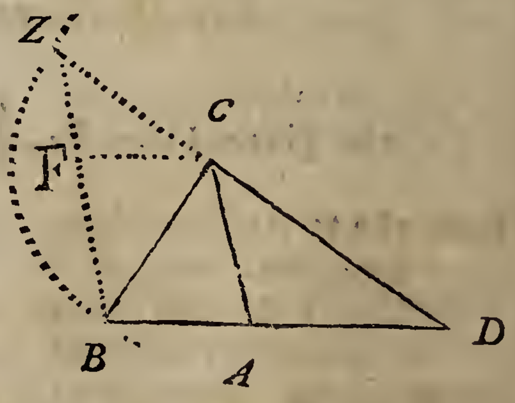


Upon the longest Line AB set off the next longest Line AD, *viz.* make $DB = AB - AD$; then upon the Point D set the other Line DC at any Angle, either right or oblique, and draw the Right-line AC, continuing it a sufficient Length; make BF parallel to DC, and it will be the fourth Proportional requir'd. That is, $AD : DC :: AB : BF$.

THEOREM XIV.

If any Angle of a plane Triangle be bisected (viz. *divided into two equal Angles*) *with a Right-line,* (viz. *as* C A *is suppos'd to do the Angle* B C D *it will cut the opposite Side* (viz. B D) *in Proportion to the other two Sides of the Triangle.* (3. *e.* 6.)

Demonstration.



Produce the Side DC, until $CZ = CB$; join the Points ZB with a Right-line, and draw the Line FC parallel to BD. Then will $\triangle CZF$, be like to $\triangle DCA$.

For $\angle ZCF = \angle D$ and $\angle Z$ is common to both $\triangle$s, consequently, $\angle ZFC = \angle CAD$ and $FC = BA$.

Therefore $BA (= FC) : BC (= ZC) :: AD : CD$. Q. E. D.

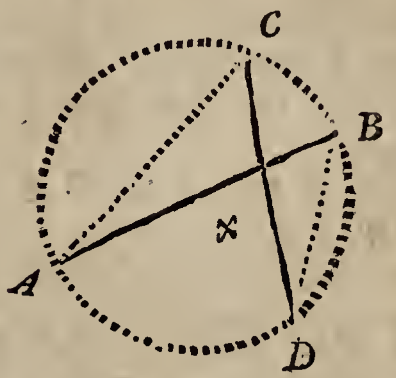
THE-

THEOREM XV.

If two Right-lines (howsoever drawn) within a Circle, do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line, (35. e. 3.)

That is, if two Lines, as AB and CD do cut each other in any Point, as at x, then will $Ax \times Bx = Dx \times Cx$.

Demonstration.

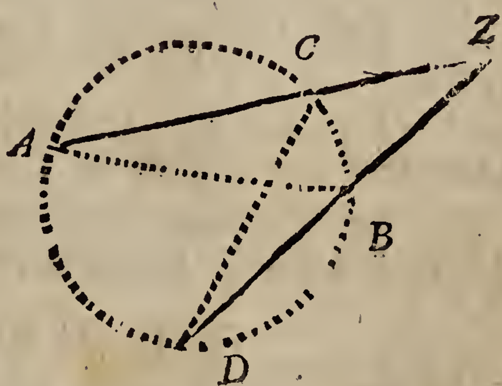


Join the Points AC and BD with Right-lines; then will the $\triangle\, CxA$ be like to the $\triangle\, BxD$. For $\angle B = \angle C$, and $\angle A = \angle D$. By Corollary to Theorem 9.
And $\angle AxC = \angle BxD$ by Theor. 2. Therefore it will be $Ax : Dx :: Cx : Bx$, By Theorem 13. Consequently, $Ax \times Bx = Dx \times Cx$. Q. E. D.

THEOREM XVI.

If two Right-lines are so drawn within a Circle, as being continued they will meet in a Point out of the Circle's Periphery, the Rectangle made of one whole Line, and its Part out of the Circle, will be equal to the Rectangle of the other whole Line, and its Part out of the Circle, (36, 37, e. 3.)

That is, if the Lines AC and DB be continued unto the Point Z;
Then will $AZ \times CZ = DZ \times BZ$.



Demonstration.

Draw the Lines AB and CD; then will $\triangle\, CZD$ be like to the $\triangle\, BZA$, for $\angle A = \angle D$, and $\angle Z$ is common to both $\triangle\,\triangle$. Consequently, $\angle ABZ = \angle DCZ$. By Theorem 4.
Therefore $AZ : BZ :: DZ : CZ$. *Ergo*, $AZ \times CZ = DZ \times BZ$.

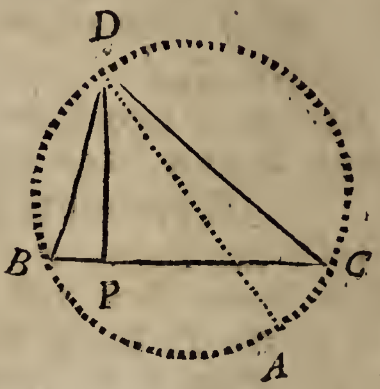
THEOREM XVII.

If from any Angle of a plane Triangle inscrib'd in a Circle, there be let fall a Perpendicular upon the opposite Side;

as

(as DP) as that Perpendicular is in Proportion to one of the Sides *including the Angle, so is the other* Side *including the Angle to the Diameter of the Circle.*

Demonstration.



Let *BCD* be the propos'd △. From the ∠ at *D* draw the Diameter *DA*; then will ∠ *A* = ∠ *B*, because they both stand upon the same Arch *DC*, and ∠ *DCA* = 90°. By Theorem 10. Consequently the ∠ *ADC* = ∠ *BDP*. By Theorem 4. Therefore △ *DCA* is like to the △ *DPB*; and therefore,

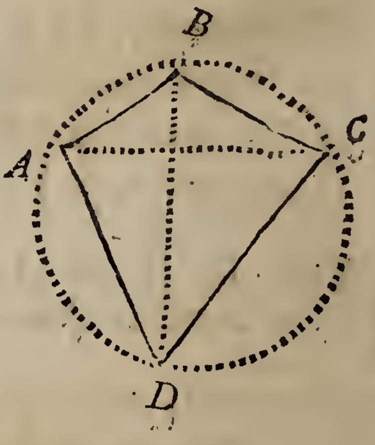
DP : *DB* :: *DC* : *DA*; Or, *DP* : *DC* :: *DB* : *DA*. Q. E. D.

THEOREM XVIII.

If any Quadrangle (that is, a Trapezium) be inscribed within a Circle; the two opposite Angles taken together are equal to two Right-angles, viz. 180°. *(22. e. 3.)*

That is, in the Quadrangle *ABCD*, the ∠ *A* + ∠ *C* = 180°. And the ∠ *B* + ∠ *D* = 180°.

Demonstration.



Draw the two Diagonals *AC* and *BD*; then will the ∠ *BDA* = ∠ *BCA*, and the ∠ *BDC* = ∠ *BAC*.
By Corol. to Theorem 9.
But ∠ *ABC* + ∠ *BCA* + ∠ *BAC* = 180°.
By Theorem 4.
And the ∠ *BDA* + ∠ *BDC* = ∠ *ADC*.
Therefore the ∠ *ABC* + ∠ *ADC* = 180°.

And by the same Way of Arguing, it may be prov'd, that the ∠ *BAD* + ∠ *BCD* = 180°. Q. E. D.

THEOREM XIX.

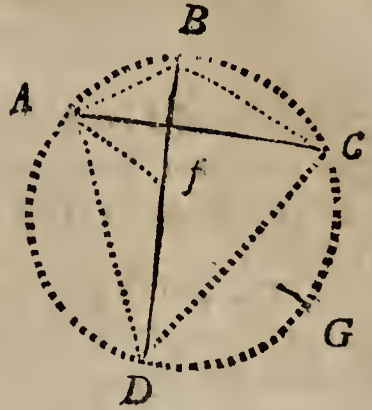
If in any Quadrangle inscrib'd within a Circle, there be drawn two Diagonals, as AC and BD, the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the opposite Sides *of the Quadrangle.*

That is, $AC \times BD = AB \times CD + AD \times BC$.

Demon-

Demonstration.

Make the Arch $DG =$ Arch BC, and from the Points AG draw the Line Af, and it will form the $\triangle$ AfD, like to the $\triangle$ ABC. For the $\angle fAD = \angle BAC$, because the Arches DG and BC are equal.



Again, the $\angle fDA = \angle BCA$ because they stand both upon the Arch AB. Consequently the $\angle AfD = \angle ABC$. By Theorem 4.

Therefore it will be $AC : BC :: AD : Df$. By Theorem 13.

$$Ergo\ \frac{BC \times AD}{AC} = Df.$$

Again, the $\triangle BAf$, and $\triangle ACD$, are alike. For $\angle ABf = \angle ACD$, and $\angle BAf = \angle CAD$, because the $\angle fAD = \angle BAC$. And the $\angle CAf$ is common to both $\triangle\triangle$. Consequently the $\angle AfB = \angle ADC$.

Therefore $AC : CD :: AB : Bf$. By Theorem 13.

$Ergo\ \dfrac{CD \times AB}{AC} = Bf$. But $Df + Bf = BD$.

Consequently, $BC \times AD + CD \times AB = BD \times AC$.

Q. E. D.

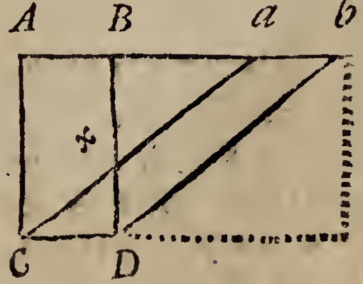
THEOREM XX.

All Parallelograms (whether right or oblique-angled) that stand upon the same Base, or upon equal Bases, and betwixt the same Parallels, are equal one to another, (35 *and* 36 *e.* 1.)

That is, $\square ABCD = \square abCD$.

Demonstration

Because $AB = CD = ab$ by Supposition; therefore $Aa = Bb$; for Ba is common to both. And because $AC = BD$, and the $\angle A = \angle B$, therefore the $\triangle ACa = \triangle BDb$; and if from both $\triangle\triangle$ there be taken the $\triangle Bxa$ common to both, there will remain the Trapeziums $ABxC = abxD$. Per Axiom 5.



But

But Trapezium $ABxC + \triangle CxD = \square ABCD$.
And Trapezium $abxD + \triangle CxD = \square abCD$.
Consequently, $\square ABCD = \square abCD$. Q. E. D.

Corollary.

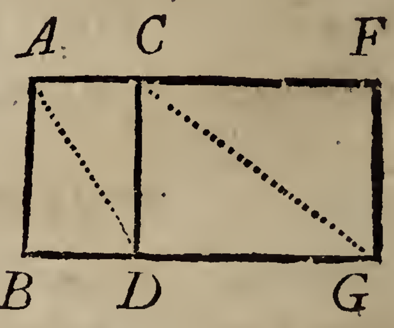
Hence it will be easy to conceive, that all Triangles which stand upon the same Base, or upon equal Bases, and between the same Parallels, (*viz.* having the same Height) are equal one to another, (37 and 38 *e.* 1.)

For all Triangles are the Halves of their circumscribing Parallelograms; and therefore, if the Wholes be equal, their Halves will also be equal.

THEOREM XXI.

Parallelograms (and consequently Triangles) which have the same Height, have the same Proportion one to another as their Bases have. (1. *e.* 6.)

Demonstration.



Draw AF parallel to BG, and draw AB, CD, FG Perpendiculars to them.

Then will $BD \times AB = \square ABCD$. And because $CD = AB$, therefore $DG \times AB = \square CDFG$. But $BD : DG :: BD \times AB : DG \times AB$.

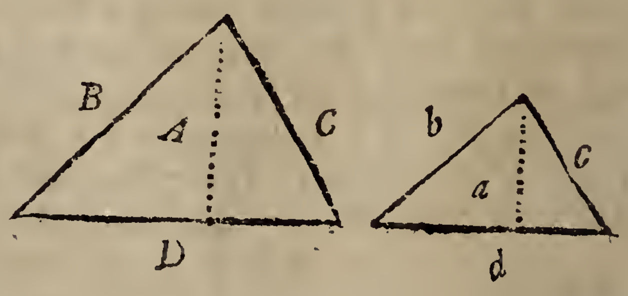
And consequently $\triangle ABD : \triangle CDG :: BD : DG$, &c. Q. E. D.

THEOREM XXII.

Like Triangles are in duplicate Ratio to that of their homologous Sides. (19. *e.* 6.)

That is, the Area's of like Triangles, are in Proportion one to another, as are the Squares of their like Sides.

Demonstration.



Suppose the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be those mark'd with the same Letters.

Let

Let A and a be Perpendiculars to the two Bases D and d.

Then $\frac{1}{2}\,D\,A =$ the Area of $\triangle\, BCD$ } by Lemma 3. Page 303.
And $\frac{1}{2}\,d\,a =$ the Area of $\triangle\, bcd$ }

But	1	$B : b :: D : d$	&c. By Theorem 13.
And	2	$B : b :: A : a$	
Conseq.	3	$D : d :: A : a$	
3 ∵	4	$Da = dA$	
$4 \times \frac{1}{2} Dd$	5	$\frac{1}{2}DDda = \frac{1}{2}DddA$.	By Axiom 3.
5, hence	6	$DD : dd :: \frac{1}{2}DA : \frac{1}{2}da$.	And so for other Sides.

Q. E. D.

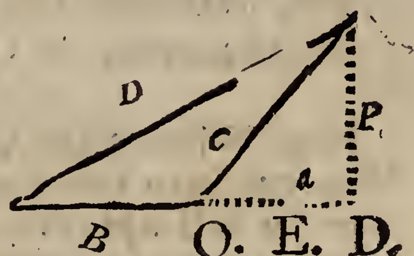
THEOREM XXIII.

In every Obtuse-angled Triangle, (as BCD) the Square of the Side subtending the obtuse Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made of one of the Sides (as B) and the Segment or Part of that Side produc'd, (as a) until it meet with the Perpendicular (P) let fall upon it. (12. e. 2.)

That is, $DD = BB + CC + 2Ba$.

Demonstration.

First	1	$DD = PP + aa + 2Ba + BB$
And	2	$CC = PP + aa$
$1 - 2$	3	$DD - CC = 2Ba + BB$
$1 + CC$	4	$DD = BB + CC + 2Ba$



Q. E. D.

Corollary.

Hence it is evident, that if the Sides of any obtuse-angled Triangle are given, the Segment (a) of the Side produc'd, or the Perpendicular (P) may be easily found.

THEOREM XXIV.

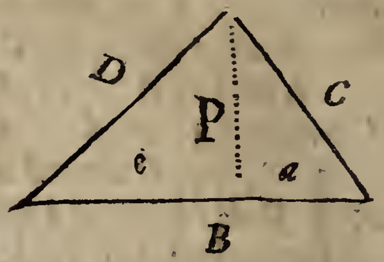
If a Perpendicular (as P) be let fall in any acute-angled Triangle, (as BCD) the Square of either of the two Sides (as D) is less than the Squares of the other Side, and that Side upon which the Perpendicular falls (viz. C and B) by a double Rectangle made of the Side B, and that Segment or Part of it (viz. a) which lies next to the Side C. (13. e. 2.)

That is, $DD + 2Ba = BB + CC$.

Demon-

Demonstration.

	First	1	$DD = PP + ee$ } By *Theo.* 11.
	And	2	$CC = PP + aa$ }
	But	3	$B - a = e$ by Figure.
3 ⊕	2	4	$BB - 2Ba + aa = ee$
4 −	aa	5	$BB - 2Ba = ee - aa$
1 −	2	6	$DD - CC = ee - aa$
5, —	6	7	$DD - CC = BB - 2Ba$
7 ∓		8	$DD + 2Ba = BB + CC$



Q. E. D.

Corollary.

Hence it follows, that if the Sides of any acute-angled Triangle be known, the Perpendicular (P) and the Segments of the Side whereon it falls, (*viz.* a. e.) may be easily found.

CHAP. IV.

The **Solution** *of several Easy* **Problems** *in Plane Geometry, whereby the Learner may (in Part) perceive the Application, or Use of the foregoing Theorems.*

Note, When a Line, or the Side of any plane Triangle, is any Way cut into two (or more) Parts, either by a perpendicular Line let fall upon it, or otherwise, those Parts are usually call'd Segments; and so much as one of those Parts is longer than the other, is call'd the Difference of the Segments.

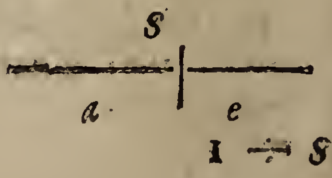
And when any Side of a Triangle, or any Segment of its Side is given, it is usually mark'd with a small Line cross it, thus —|— and those Sides, or Parts of Sides, that are sought, are mark'd with four Points, thus —∺—

PROBLEM I.

To cut or divide a given Right-line (as S) into extreme and mean Proportion, (11. e. 2.)

THat is, to divide a Line so, that the Square of the greater Segment (or Part) a, may be equal to the Rectangle made of the whole Line S, and the lesser Segment e.

Viz.	1	$Se = aa$, by the Problem.
And	2	$S - a = e$, for $S = a + e$



$1 \div S$	3	$\frac{aa}{S} = e$
2 and 3	4	$\frac{aa}{S} = S - a$. By *Axiom* 5.
$4 \times S$	5	$aa = SS - Sa$
$5 + Sa$	6	$aa + Sa = SS$
6, Solved	7	$a = \sqrt{SS + \frac{1}{4} SS:} - \frac{1}{2} S$. See *Pages* 195. 196.

Note, *The last Problem cannot be truly answer'd by Numbers; but Geometrically it may be perform'd thus.*

1. Make a Square; whose Side is $= S$ the given Line, and bissect one of its Sides in the Middle, as at C; upon the Point C describe such a Semi-circle, as will pass through the remotest Points of the Square, and complete its Diameter.

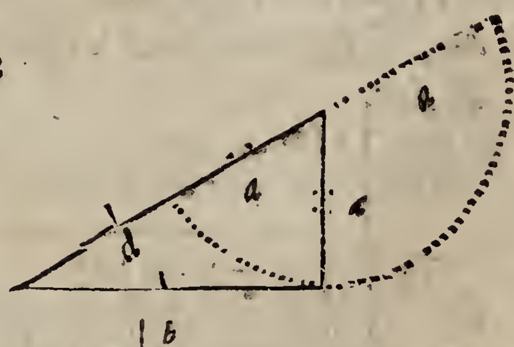
2. Then will either Part of the Diameter, on each End of the Side S, be $= a$, the greater Segment sought.

But $a + S : S :: S : a$. By *Theorem* 13.
Ergo, $aa + Sa = SS$. Which was to be done.

PROBLEM II.

The Base of any Right-angled Triangle, and the Difference between the Hypothenuse and Cathetus or Perpendicular being given, to find the Cathetus, &c.

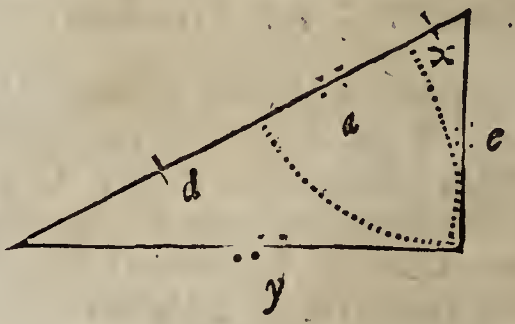
Let	1	$b = 72$
	2	$d = 32$
And	3	$a =$ Cathetus sought
Then	4	$bb + aa = dd + 2da + aa$ By *Theorem* 11.
$4 - aa$	5	$bb = dd + 2da$
$5 - dd$	6	$2da = bb - dd$
$6 \div 2d$	7	$a = \frac{bb - dd}{2d} = 65$
Or	8	$b : d + 2a :: d : b$. By *Theorem* 13.
8 ∴	9	$bb = dd + 2da$. As before at the 5th Step.



Here you see, that either Way raises the same Æquation; neither is there any constant Method or Road to be observ'd in solving Geometrical Problems, but every one makes use of such Ways and *Theorems* as happen to come first into their Mind, the Result being every Way the same.

PROBLEM III.

The Difference between the Base and Hypothenuse of any Right-angled Triangle, and the Difference between the Cathetus and Hypothenuse being both given, to find the Triangle.



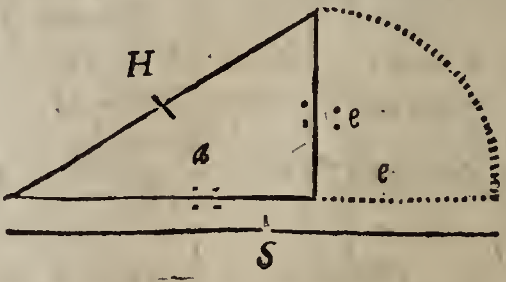
Let	1	$d = 32$
	2	$x = 25$
And	3	$d + x + a =$ *Hypot.*
Then	4	$d + a = y$ by the *Prob.*
	5	$x + a = e$ by the *Prob.*
4 ⊛ 2	6	$dd + 2da + aa = yy$
5 ⊛ 2	7	$xx + 2xa + aa = ee$
3 ⊛ 2	8	$dd + 2dx + 2da + 2xa + xx + aa = \square$ Hypothenuse.
6 + 7	9	$dd + 2da + 2xa + xx + 2aa = yy + ee$

The two last Steps are equal; by *Theorem* 11. Consequently, if those Things that are equal in both be taken away, the Remainders will be equal. By *Axiom* 2.

That is,	10	$aa = 2dx = 1600$
10 ɯ 2	11	$a = \sqrt{2dx} = 40$
1 + 11	12	$d + a = 72 = y$ The Base.
2 + 11	13	$x + a = 65 = e$ The Cathetus.
1 + 2 + 11	14	$d + x + a = 97$ The Hypothenuse.

PROBLEM IV.

The Hypothenuse, and the Sum of the other two Sides of any Right-angled Triangle being given, thence to find the Sides.



Let	1	$H = 97$
And	2	$a + e = S = 137$
By Fig.	3	$aa + ee = HH$
2 ⊛ 2	4	$aa + 2ae + ee = SS$
4 — 3	5	$2ae = SS - HH$
3 — 5	6	$aa - 2ae + ee = 2HH - SS$
6 ɯ 2	7	$a - e = \sqrt{2HH - SS}$

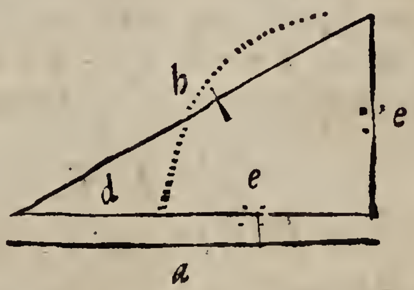
2 + 7

2 + 7	8	$2a = s + \sqrt{2HH - ss} = 144$
8 ÷ 2	9	$a = \frac{s + \sqrt{2HH - ss}}{2} = 72$. The Base requir'd.
2 − 9	10	$e = \frac{s - \sqrt{2HH - ss}}{2} = 65$. The Cathetus.

PROBLEM V.

The Hypothenuse, and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.

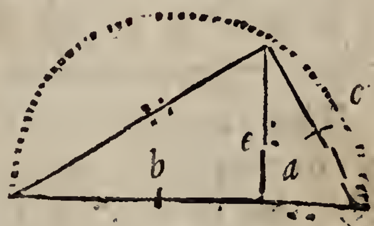
Let	1	$b = 97$ As before.
And	2	$a - e = d = 7$ Quere a.
By Fig.	3	$aa + ee = bb$
2 @ 2	4	$aa - 2ae + ee = dd$
3 − 4	5	$2ae = bb - dd$
3 + 5	6	$aa + 2ae + ee = 2bb - dd$
6 ɯ 2	7	$a + e = \sqrt{2bb - dd}$
2 + 7	8	$2a = d + \sqrt{2bb - dd} = 144$
8 ÷ 2	9	$a = 72$
7 − 2	10	$2e = \sqrt{2bb - dd} : - d = 130$
1 ÷ 2	11	$e = 65$.



PROBLEM VI.

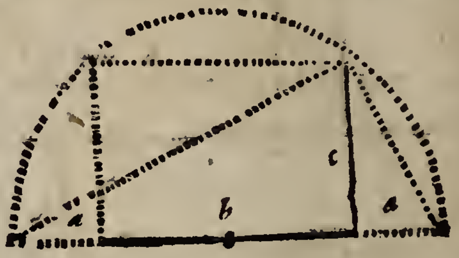
In any Right-angled Triangle, either the Base, or Cathetus, and the Alternate Segment of the Hypothenuse (made by a Perpendicular let fall from the Right Angle) being given, to find the other Segment.

Let	1	$c = 45$ The Cathetus.
And	2	$b = 48$ The alternate Seg.
Then	3	$b : e :: e : a$ Quere a
3 ∴	4	$ba = ee$
Again	5	$cc - aa = ee$ By *Theor.* 11.
4, 5	6	$ba = cc - aa$
6 + aa	7	$aa + ba = cc$
7, C □	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$
8 ɯ 2	9	$a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb}$
9 − $\frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} : - \frac{1}{2}b = 27$ And so on for e, &c.



I shall now shew the Geometrical Construction, or Solution of the three Cases of Quadratic Æquations, promis'd in Page 202. Let the first Example be that above, *viz.* $aa + ba = cc$. Case 1.

Make the Co-efficient b, and the Root of the Resolvend, (which is here) c, into a Right-angled Parallelogram. Per *Problem* 13. *Chap.* 2.

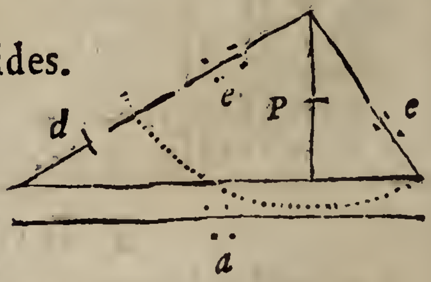


And upon the middle Point of the Side $= b$ describe such a Semicircle, as will pass through the remotest Points (or Angles) of the Parallelogram, completing its Diameter, as in the annexed Scheme. Then will either Part of the Diameter on each End b, be equal to a, the other Part will be $a + b$, and the Side c will be a mean Proportional between them. That is, $a + b : c :: c : a$. By *Theorem* 13.

Consequently $aa + ba = cc$. Which was to be done.

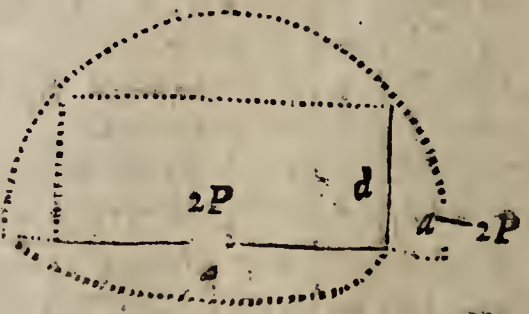
PROBLEM VII.

The Difference between the Base and Cathetus of any Right-angled Triangle, and the Perpendicular let fall from the Right Angle upon the Hypothenuse, being given; Thence to find the Hypothenuse, &c.



Let	1	$d = 15$ The Differ. of the Sides.
And	2	$p = 36$
Quere a	3	$a =$ The Hypothenuse.
By Fig.	4	$d + e : p :: a : e$
4 ∵	5	$de + ee = pa$
Again	6	$dd + 2de + 2ee = aa$. By *Theorem* 11.
5 × 2	7	$2de + 2ee = 2pa$
6 − 7	8	$dd = aa - 2pa$ *Case* 2.
8 C □	9	$aa - 2pa + pp = dd + pp = 1521$.
9 ω 2	10	$a - p = \sqrt{dd + pp} = 39$
10 + p	11	$a = p + \sqrt{dd + pp} = 75$, &c. for e per *Step* 5.

The Geometrical Construction of this *Case* 2. *viz.* $aa - 2pa = dd$. may be perform'd in the very same Manner as the last Case was; that is, by making a Right-angled Parallelogram of the Co-efficient $2p$, and the $\sqrt{dd}$, *viz.* d, &c. as in the annexed Figure.



Then

Then will the Greater Part of the Diameter to one End of the Parallelogram be $=a$, and the lesser Part will be $a-2p$.

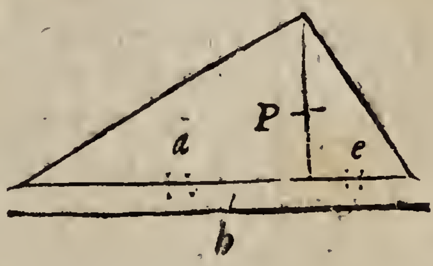
For $a:d::d:a-2p$. By *Theorem* 13.

Consequently, $aa-2pa=dd$. Which was to be done.

PROBLEM VIII.

The Hypothenuse of any Right-angled Triangle, and the Perpendicular let fall from the Right Angle upon the Hypothenuse, being given, To find the Greater Segment of the Hypothenuse, &c.

Let	1	$h=75$ The Hypothenuse
And	2	$p=36$
Then	3	$a+e=h$ *Quere* a
per Fig.	4	$a:p::p:e$
4 ∵	5	$\frac{pp}{a}=e$
$3-a$	6	$h-a=e$
5, 6	7	$h-a=\frac{pp}{a}$
$7\times a$	8	$ha-aa=pp$ *Case* 3.
$8\pm$	9	$aa-ha=-pp$
9 C□	10	$aa-ha+\frac{1}{4}hh=\frac{1}{4}hh-pp=110,25$
10 ω 2	11	$a-\frac{1}{2}h=\sqrt{\frac{1}{4}hh-pp}=10,5$
$11+\frac{1}{2}h$	12	$a=\frac{1}{2}h\pm\sqrt{\frac{1}{4}hh-pp}=48$. Or $a=27$



The Geometrical Construction of *Case* 3. *viz.* $ha-aa=pp$, may be thus perform'd. Draw a Right Line (of any convenient length at Pleasure) and near its Middle erect a Perpendicular $=p$. *viz.* of the same length with the Root of the Resolvend. From the Top Point or upper End of that Perpendicular, set off half the Length of the Co-efficient, *viz.* $\frac{1}{2}h$, and upon the Point where $\frac{1}{2}h$ just touches the first Line (with the same Distance) describe a Semi-circle; then will its Diameter h be cut by the Perpendicular p into two Segments, which are the two Values of the Root a, *viz.* the greater and lesser Roots, both taken together being always equal to the Co-efficient, (vide *Page* 201.)

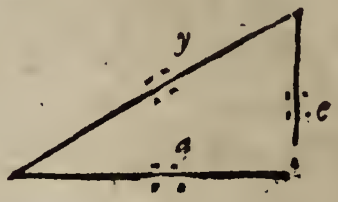
For $h-a:p::p:a$ by *Theorem* 13.

Ergo $ha-aa=pp$. Which was to be done.

PROBLEM

PROBLEM IX.

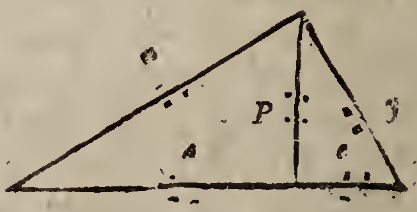
The Perimeter, viz. the Sum of all the three Sides of any Right-angled Triangle, and its Area being given, thence to find each Side.



Viz. Let	1	$a+e+y=s=234$ The Sum of the Sides.
And	2	$\frac{1}{2}ae=A$ The Area $=2340$
Again	3	$aa+ee=yy$ By Figure
2×4	4	$2ae=4A$
$3+4$	5	$aa+2ae+ee=yy+4A$
$1-y$	6	$a+e=s-y$
6 ⊚ 2	7	$aa+2ae+ee=ss-2sy+yy$
5, 7	8	$yy+4A=ss-2sy+yy$
8 ±	9	$2sy=ss-4A=45396$
$9 \div 2s$	10	$y=\frac{ss-4A}{2s}=\frac{1}{2}s-\frac{2A}{s}=97$ The Hypothenuse
6, 10	11	$a+e=s-y=137$
$3-4$	12	$aa-2ae+ee=yy-4A=49$
12 ω 2	13	$a-e=\sqrt{49}=7$
$11+13$	14	$2a=137+7=144$
$13 \div 2$	15	$a=72$ The Base.
$11-15$	16	$e=137-72=65$ The Cathetus.

PROBLEM X.

In any Right-angled Triangle a Perpendicular being let fall from the Right Angle upon the Hypothenuse, if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.



Viz. If	1	$a+u=s=108$
And	2	$e+y=z=72$
To find		$a,\ e,\ u,\ y.$ and p
$1-a$	3	$u=s-a$
3 ⊚ 2	4	$uu=ss-2sa+aa$
$4-aa$	5	$uu-aa=ss-2sa=pp$
$2-e$	6	$z-e=y$
6 ⊚ 2	7	$zz-2ze+ee=yy$
$7-ee$	8	$zz-2ze=yy-ee=pp$
5, 8.	9	$zz-2ze=ss-2sa$
By Fig.	10	$a:p::p:e$
10 ∷	11	$ae=pp$
5, 11	12	$ae=ss-2sa$

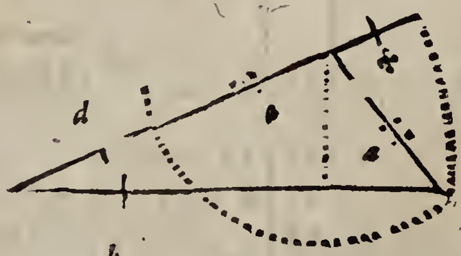
12 ÷ a

$12 \div a$	13	$e = \frac{ss - 2sa}{a}$
$13 \times 2z$	14	$2ze = \frac{2zss - 4zsa}{a}$
$9 + 14$	15	$zz = ss - 2sa + \frac{2zss - 4zsa}{a}$
$15 \times a$	16	$zza = ssa - 2saa + 2zss - 4zsa$
$16 \pm$	17	$2saa + zza + 4zsa - ssa = 2zss$
$17 \div 2s$	18	$aa + \frac{zza}{2s} + 2za - \frac{1}{2}sa = zs$
Substitute	19	$2x = \frac{zz}{2s} + 2z - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = zs = 7776$
20 C □	21	$aa + 2xa + xx = zs + xx = 11025$
21 ω 2	22	$a + x = \sqrt{zs + xx} = 105$
$22 - x$	23	$a = \sqrt{xs + xx} : - x = 48$
$1 - 23$	24	$u = 60 =$ The Base.
per 13	25	$e = \frac{ss}{a} - 2s = 27$
$2 - 25$	26	$y = 45 =$ the Cathetus.
$23 + 25$	27	$a + e = 75 =$ the Hypothenuse.

PROBLEM XI.

The Difference of the Sides of any Oblique-angled plane Triangle, the Difference of the Segments of the Base, and the Difference between the greater Side, and the Base, being given, to find the Base, &c.

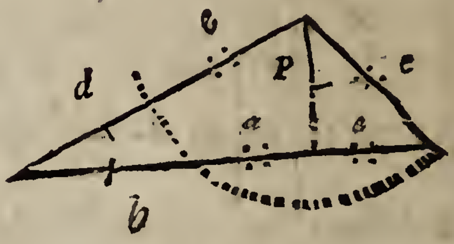
Let	1	$d =$ the Difference of the Sides $= 405$.
	2	$b =$ the Difference of the Segments $= 495$
	3	$x = 165$ the Difference of the greater Side and Base.
And	4	$a =$ the least Side
Then	5	$d + a + x =$ the Base
And	6	$\overline{d + a + x} : \overline{d + 2a} :: d : b$ By *Theorem* 16.
6 ∵	7	$db + ba + bx = dd + 2da$
$7 \pm$	8	$2da - ba = db + bx - dd$
$8 \div 2d - b$	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$
$1 + 9$	10	$d + a = 780 =$ the greatest Side.
$3 + 10$	11	$d + a + x = 945 =$ the Base.



PROBLEM

PROBLEM XII.

The Difference of the Sides of any plane Triangle; the Difference of the Segments of the Base, and the Perpendicular let fall from the vertical Angle, being given, thence to find all the Sides.



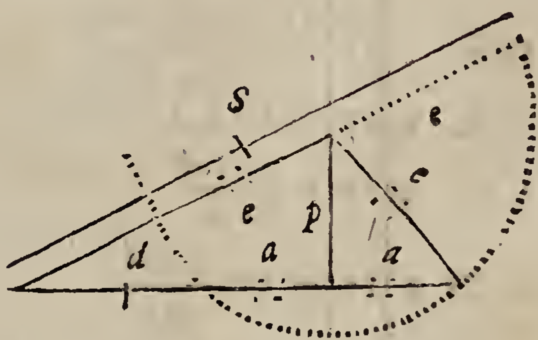
Let	1	$d=405$ as before.
	2	$b=495$
And	3	$p=300$
Quere	4	$a=$ the least Segment.
Then	5	$b+2a : d+2e :: d : b$
5 ⁘	6	$bb+2ba=dd+2de$
6 − dd	7	$bb-dd+2ba=2de$
Substitute	8	$2x=bb-dd=81000$
7, 8	9	$2x+2ba=2de$
÷ $2d$	10	$\frac{x+ba}{d}=e$
But	11	$pp+aa=ee$ By *Theorem* 11.
10 ⊛ 2	12	$\frac{xx+2xba+bbaa}{dd}=ee$
11, 12	13	$\frac{xx+2xba+bbaa}{dd}=pp+aa$
13 × dd	14	$xx+2xba+bbaa=ppdd+ddaa$
14 ±	15	$bbaa-ddaa+2xba=ppdd-xx$
8, 15	16	$2xaa+2xba=ppdd-xx$
16 ÷ $2x$	17	$aa+ba=\frac{ppdd}{2x}-\frac{1}{2}x$
17 C □	18	$aa+ba+\frac{1}{4}bb=\frac{1}{4}bb+\frac{ppdd}{2x}-\frac{1}{2}x$
18 ω 2	19	$a+\frac{1}{2}b=\sqrt{\frac{1}{4}bb+\frac{ppdd}{2x}-\frac{1}{2}x}$
19 − $\frac{1}{2}b$	20	$a=\sqrt{\frac{1}{4}bb+\frac{ppdd}{2x}-\frac{1}{2}x}:-\frac{1}{2}b=225$
20 × 2	21	$2a=450$
2 + 21	22	$b+2a=945$ the Base.
10, *Num.*	23	$e=375=$ the lesser Side.
1 + 23	24	$d+e=780=$ the greater Side.

PROBLEM XIII.

The Sum of the two Sides of any plane Triangle, the Difference of the Segments of the Base, and the Perpendicular let

let fall from the Vertical Angle upon the Base, being given, thence to find the Base, and the Sides.

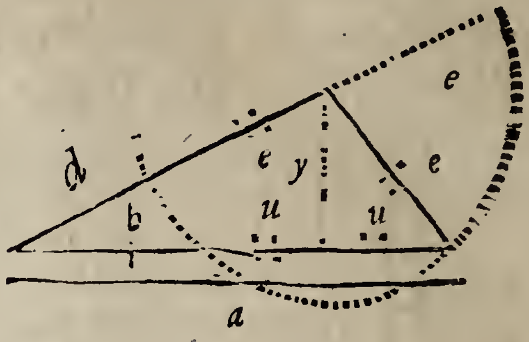
Let	1	$s = 1155$ the Sum of the Sides.
	2	$d = 495$ the Difference of the Segments.
	3	$p = 300$ the Perpendicular.
Put	4	$a =$ the least Segment.
	5	$e =$ the least Side.
Then	6	$d + 2a =$ the Base.
And	7	$s - 2e =$ the Difference of the Sides.
Per *Fig.*	8	$d + 2a : s :: s - 2e : d$
	9	$aa + pp = ee$
9 ɯ 2	10	$\sqrt{aa + pp} = e$
8 ⁘	11	$dd + 2da = ss - 2se$
11 ±	12	$2se = ss - dd - 2da$
Suppose	13	$2x = ss - dd$
Then	14	$2se = 2x - 2da$
14 ÷ 2s	15	$e = \frac{x - da}{s}$
10; 15	16	$\frac{x - da}{s} = \sqrt{aa + pp}$
16 ⊕ 2	17	$\frac{xx - 2xda + ddaa}{ss} = aa + pp$
17 × ss	18	$xx - 2xda + ddaa = ssaa + sspp$
18 ±	19	$ssaa - ddaa + 2xda = xx - sspp$
13, 19	20	$2xaa + 2xda = xx - sspp$
20 ÷ 2x	21	$aa + da = \frac{1}{2}x - \frac{sspp}{2x}$, *&c.* As before.
21, hence	22	$a = 225$
22 × 2	23	$2a = 450$
2 + 23	24	$d + 2a = 945$ the Base.
10. *Numb.*	25	$e = 375$ the lesser Side.
1 − 25	26	$s - e = 780$ the greater Side.



PROBLEM XV.

The Area of any oblique-angled Triangle; the Difference of the Sides, and the Difference of the Segments of the Base being given, thence to find the Base, &c.

Let	1	$A = 141750 =$ the Area.
	2	$d = 405$
	3	$b = 495$



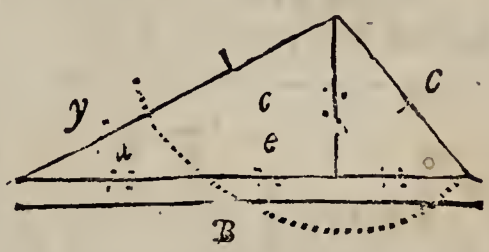
Put {	4	$y =$ the Perpendicular.
	5	$a =$ the Base.
Then	6	$\frac{1}{2}ya = A$
Per Fig.	7	$a : d + 2e :: d : b$
7 ∴	8	$ba = dd + 2de$
8 — dd	9	$ba - dd = 2de$
9 ⊛ 2	10	$bbaa - 2ddba + dddd = 4ddee$
Per Fig	11	$\frac{a - b}{2} = u$ the lesser Segment of the Base.
11 ⊛ 2	12	$\frac{aa - 2ba + bb}{4} = uu$
6 × 2	13	$ya = 2A$
13 ÷ a	14	$y = \frac{2A}{a}$
14 ⊛ 2	15	$yy = \frac{4AA}{aa}$
Per Fig.	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10 ÷ 4dd	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18 × aa	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bbaa}{4}$
19 × 4dd	20	$\left\{ \begin{array}{l} bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4 \\ - 2ddba^3 + ddbbaa \end{array} \right.$
20 ±	21	$bba^4 - dda^4 + d^4aa - ddbbaa = 16AAdd$
21 ÷	22	$aaaa - ddaa = \frac{16AAdd}{bb - dd}$
22 C □	23	$aaaa - ddaa + \frac{1}{4}dddd = \frac{16AAdd}{bb - dd} + \frac{1}{4}d^4$
23 ω 2	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$
24 + ½dd	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}$
25 ω 2	26	$a = \sqrt{:\frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}} = 945.$

PROBLEM

PROBLEM XV.

There is an oblique-angled plane Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base, the least Side, and the Base are given, And the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the Base: 'Tis requir'd to find the Segments of the Base, &c.

Let	1	$c = 56 =$ the least Side.
	2	$B = 92 =$ the Base.
And	3	$a + 2e = B$
Put	4	$y =$ the Difference of the Sides.
Then	5	$cy = aa$ by the Question.
By *Figure.*	6	$B : 2c + y :: y : a$, for $B = a + 2e$
6 ⁘	7	$Ba = 2cy + yr$
5 × 2	8	$2cy = 2aa$
7 — 8	9	$Ba - 2aa = yy$
5 ⊚ 2	10	$ccyy = aaaa$
10 ÷ cc	11	$yy = \frac{aaaa}{cc}$
9 11	12	$Ba - 2aa = \frac{aaaa}{cc}$
12 × cc	13	$ccBa - 2ccaa = aaaa$
13 ÷ a	14	$ccB - 2cca = aaa$
14 + $2cca$	15	$aaa + 2cca = ccB$
15 *in Num*	16	$aaa + 6272a = 288512$

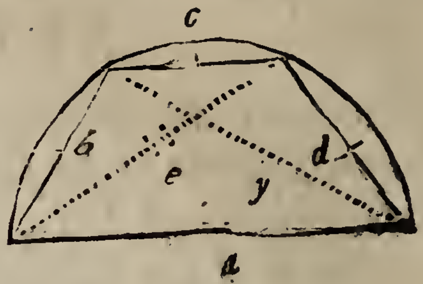


The Value of a, in this Æquation, may be found as in the Examples, *Page* 238, *viz.* by putting $r + e = a$, &c. As in those Examples, you will find $a = 37,55501$, &c.

PROBLEM XVI.

The three Chords or Subtenses of three Arches completing a Semi-circle, being each given, thence to find the Diameter of that Circle.

That is, *any Trapezium being inscrib'd in a Semi-circle, if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter, or fourth Side.*

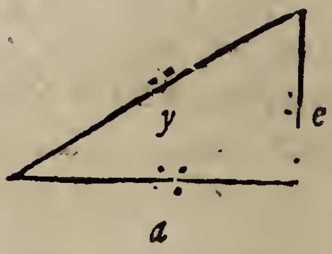


Let	1	$b = 3$ the three Sides.
	2	$c = 4$
	3	$d = 5$
Quere	4	$a =$ the Diameter sought.
		Draw the two Diagonals e and y
Then	5	$ca + bd = ey$. By Theorem 19.
And	6	$aa - bb = yy$ By Theorem 10 and 11
	7	$aa - dd = ee$
5 ⊚ 2	8	$ccaa + 2bdca + bbdd = eeyy$
6 × 7	9	$aaaa - bbaa - ddaa + bbdd = eeyy$
8, = 9	10	$aaaa - bbaa - ddaa = ccaa + 2bdca$
10 ÷ a	11	$aaa - bba - dda = cca + 2bdc$
11 − caa	12	$aaa - bba - dda - cca = 2bdc$
12, Num.	13	$aaa - 50a = 120$

This Æquation being solv'd, as in *Example* 2, *Page* 240, you will find $a = 8{,}05581$, &c.

PROBLEM XVII.

In any Right-angled Triangle, the Area and the Sum of the Hypothenuse, when added to either Side, being given, thence to find the Sides, &c.



Suppose	1	$\frac{ae}{2} = A = 1350$. The Area
	2	$y + e = s = 120$. The Sum, &c.
	3	Quere a, e, and y
1 × 2	4	$ae = 2A$
4 ÷ a	5	$e = \frac{2A}{a}$
Per Fig.	6	$aa + ee = yy$
2 − e	7	$y = s - e$
5, 7	8	$y = s - \frac{2A}{a}$
8 ⊚ 2	9	$yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
5 ⊚ 2	10	$ee = \frac{4AA}{aa}$
10 + aa	11	$aa + ee = \frac{4AA}{aa} + aa$

6, 9, 11

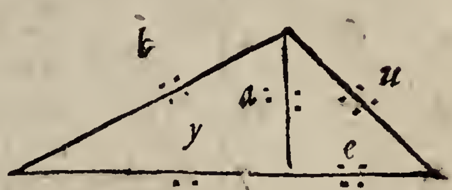
6, 9, 11	14	$\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
12, That is	13	$aa = ss - \frac{4sA}{a}$
13 $\times$ a	14	$aaa = ssa - 4sA$
14 $\pm$	15	$ssa - aaa = 4sA$
15, in *Num.*	16	$14400a - aaa = 648000$

The Value of a, in this Æquation, may be found as in the third *Example*, *Page* 241: That is, by making $r + e = a$, &c. it will be found that $a = 60$.

PROBLEM XVIII.

There is an oblique-angled plane Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base, the Sum of each Segment of the Base, when added to its adjacent or next Side, and the Area of the Triangle, are given, To find the Perpendicular, and each Side.

Let	1	$y + b = z = 1500$ } *Quere* y, b, e and u
	2	$e + u = s = 600$
	3	$A =$ the Area $= 141750$
And	4	$a =$ the Perpendicular sought.
Then	5	$\overline{y + e} \times \frac{1}{2}a = A$
$5 \times \overline{2} \div a$	6	$y + e = \frac{2A}{a}$
Per Fig.	7	$yy + aa = bb$
	8	$ee + aa = uu$
1 — y	9	$b = z - y$
2 — e	10	$u = s - e$
9 ⓖ 2	11	$bb = zz - 2zy + yy$
10 ⓖ 2	12	$uu = ss - 2se + ee$
7, 11	13	$zz - 2zy = aa$
8, 12	14	$ss - 2se = aa$
13 $\pm$	15	$zz - aa = 2zy$
14 $\pm$	16	$ss - aa = 2se$
15 $\div$ 2z	17	$\frac{zz - aa}{2z} = y$
16 $\div$ 2s	18	$\frac{ss - aa}{2s} = e$
17 + 18	19	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$



Having found the Value of a, from the 24th Step, e and y will be easily found by these 2 Steps. And b, u by the 9th and 10th Steps.

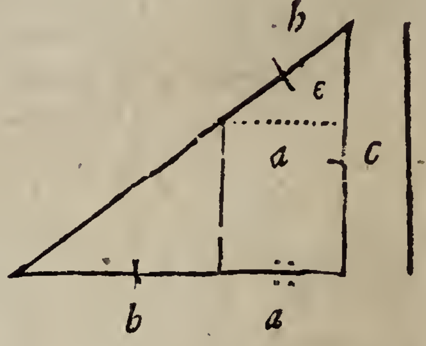
6, 19

6,	19	20	$\frac{zz-aa}{2z}+\frac{ss-aa}{2s}=\frac{2A}{a}$
20 ×	2z	21	$zz-aa+\frac{zss-zaa}{s}=\frac{4zA}{a}$
21 ×	s	22	$zzs-saa+zss-zaa=\frac{4zAs}{a}$
22 ×	a	23	$zzsa-saaa+zssa-zaaa=4zAs$
23, *Num.*		24	$900000a-aaa=243000000$
			Here $a=300$ found as in the last *Problem.*

PROBLEM XIX.

There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; there is given the Cathetus, that Segment of the Hypothenuse next to the Cathetus, and the alternate Segment of the Base, thence to find the Base, &c.

Viz. Let		1	$b=20$. $c=24$. and $h=15$
Then		2	$b+a=$ the Base. *Quere* a
Here		3	$b+a:c::a:e$ Per *Figure.*
And		4	$aa+ee=hh$ Per *Figure.*
3	∵	5	$\frac{ca}{b+a}=e$
5 ⊛	2	6	$\frac{ccaa}{bb+2ba+aa}=ee$
4 −	aa	7	$hh-aa=ee$
6,	7	8	$\frac{ccaa}{bb+2ba+aa}=hh-aa$
8 ×		9	$ccaa=hhbb-bbaa+2hhba-2ba^3+hhaa-a^4$
9 ±		10	$a^4+2baaa+ccaa+bbaa-hhaa-2hhba=hhbb$
That is		11	$aaaa+40aaa+751aa-9000a=90000$



For a Solution of this Æquation, let it be made

$aaaa+baaa+caa-da=G$ *Viz.* $\begin{cases} b=40 \;.\; c=751 \\ d=9000 \;.\; G=90000 \end{cases}$

Put $r+e=a$

$$\text{Then}\left\{\begin{array}{l} r+4rrre+6rree=a^4 \\ brrr+3brre+3bree=baaa \\ crr+2cre+cee=caa \\ -dr-de=-da \end{array}\right\}=G=90000$$

Let $r=10$

Then

$$\text{Then}\left\{\begin{array}{l} +10000+4000e+600ee \\ +40000+12000e+1200ee \\ +75100+15020e+751ee \\ -90000-9000e \end{array}\right\}=G=90000$$

That is, $35100+22020e+2551ee=90000$
Hence it will be $22020e+2551ee=54900$
Consequently, $8{,}63e+ee=21{,}52=D$

$$\text{And}\ \frac{D}{8{,}63+e}=e$$

Operation	$8{,}63$)	$\dot{2}1{,}\dot{5}2$	($2{,}1=e$
	$+e=2{,}1$	20	
1 Divisor	$=10$	$1{,}52$	First $r=10$
2 Divisor	$=10{,}7$	$1{,}07$	$+e=2{,}1$
		45 &c.	

$r+e=12{,}1=r$ for a second Operation, which being involv'd, and multiply'd into the Co-efficients as before, will produce these Numbers.

$$\left.\begin{array}{l} +21435{,}8881+7086{,}24e+878{,}46ee \\ +70862{,}4400+17569{,}20e+1452{,}00ee \\ +109953{,}9100+18174{,}20e+751{,}00ee \\ -108900{,}0000-9000{,}00e \end{array}\right\}=G$$

Viz. $93352{,}2381+33829{,}64e+3081{,}46ee=90000$

Here because $93352{,}2381>90000$. Therefore $12{,}1>a$, and therefore it must needs be made $r-e=a$, which will produce the same Numbers, only all the second Signs must be chang'd.

Thus, $93352{,}2381-33829{,}64e+3081{,}46ee=90000$ from whence will arise this Æquation,

$+33829{,}64e-3081{,}46ee=3352{,}2381$

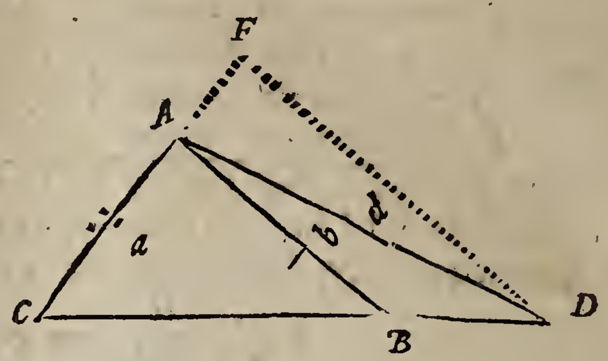
Consequently, $10{,}9784e-ee=1{,}08787332=D$

Operation	$10{,}9784$)	$1{,}\dot{0}8\dot{7}8\dot{7}3\dot{3}2$	($0{,}0999=e$
	$-e=,0999$	9792	
1. Divisor	$10{,}88$	108673	Last $r=12{,}1$
2. Divisor	$10{,}879$	97911	$-e=0\ 0999$
3. Divisor	$10{,}8785$	1076232	$r-e=12{,}0001=a$
		979065	
		&c.	

PROBLEM XX.

In the oblique-angled Triangle CAD, there is given the Side AD, and the Sum of the Sides AC+CD; also within the Triangle there is given the Line AB perpendicular to the Side CA, thence to find the Side CA,&c.

Let



Let	1	$CA + CD = s = 51$
	2	$AD = d = 32$
	3	$AB = b = 21$
And	4	$CA = a$ sought.
Then	5	$s - a = CD$
Suppose		the Line DF Parallel to AB, and CA produc'd to F.
Then		$\triangle CAB$, and $\triangle CFD$ will be alike.
And	6	$BC : CA :: DC : CF$.
But	7	$BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD = y$
6, 7	8	$\sqrt{bb + aa} : a :: s - a : a + e$
8 ∴	9	$\dfrac{sa - aa}{\sqrt{: bb + aa}} = a + e$
5 ⊚ 2	10	$ss - 2sa + aa = \square CD$
Per *Fig.*	11	$ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$
11 − aa	12	$ss - 2sa = 2ae + ee + yy$
But	13	$dd = ee + yy = \square AF + \square FD$
12 − 13	14	$ss - 2sa - dd = 2ae$
Let	15	$2x = ss - dd$
14, 15	16	$x - sa = ae$
16 ÷ a	17	$\dfrac{x - sa}{a} = e$
17 + a	18	$\dfrac{x - sa + aa}{a} = a + e$
9 ⊚ 2	19	$\dfrac{ssaa - 2saaa + a^4}{bb + aa} = \square \overline{a + e}$
18 ⊚ 2	20	$\dfrac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square \overline{a + e}$
18, 19	21	$\dfrac{ssaa - 2saaa + a^4}{bb + aa} = \dfrac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa}$

This Æquation being brought out of the Fractions, and into Numbers will become,

$- 2018aaaa + 125409aaa - 2464230{,}25aa + 35468307a =$
$= 274183922{,}25$

which being divided by 2018, the Co-efficient of the highest Power of a will

be

be $\left\{\begin{array}{l} -a^4 + 62{,}1459\,a^3 - 1221{,}125\,aa + 17575{,}9697\,a \\ = 135869{,}138875, \text{ \&c.}\end{array}\right.$

And from hence the Value of a may be found, as in the laſt *Problem*, due Regard being had to the Signs of every Term.

This Work of reducing, or preparing Æquations for a Solution by Diviſion, hath always been taught both by the Antient and Modern Writers of *Algebra*, as a Work ſo neceſſary to be done, that they do not ſo much as give a Hint at the Solution of any adfected Æquation without it.

Now it very often happens, that in dividing all the Terms of an Æquation, ſome of their Quotients will not only run into a long Series, but alſo into imperfect Fractions, (as in this Æquation above) which renders the Solution both tedious and imperfect.

To remedy that Imperfection, I ſhall here ſhew how this Æquation (and conſequently any other) may be reſolv'd without ſuch Diviſion, or Reduction.

Let $b = 2018$. $c = 125409$. $d = 2464230{,}25$.
$f = 35468307$. And $G = 274183922{,}25$

Then the precedent Æquation will ſtand thus,

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$. As before.

$$\text{Then will } \left\{\begin{array}{llll} -br^4 & -4brrre & -6brree & = -ba^4 \\ +cr^3 & +3crre & +3cree & = +ca^3 \\ -drr & -2dre & -dee & = -daa \\ +fr & +fe & \ldots\ldots & = +fa \end{array}\right\} = G.$$

This is plain and eaſily conceiv'd; The next thing will be how to eſtimate the firſt Value of r; and to perform that, let G be divided by b, only ſo far as to determine how many Places of whole Numbers there will be in the Quotient; conſequently, how many Points there muſt be, (according to the Height of the Æquation.

$$\begin{array}{l} \text{Thus } b = 2018)\; G = 274183922{,}25 \;(\; 130000 \\ \qquad\qquad\qquad 2018 \\ \qquad\qquad\qquad \overline{} \\ \qquad\qquad\qquad 7238 \text{ \&c.} \end{array}$$

Now from hence one may as eaſily gueſs at the Value of r, as if all the Terms had been divided. That is, I ſuppoſe $r = 10$, which being involv'd, *&c.* as the Letters above direct will be

$$\left.\begin{array}{l} -\ 20180000 - 80720000e - 12108000ee \\ +\ 125409000 + 376227000e + 3762270ee \\ -\ 246423025 - 492846050e - 2464230{,}25ee \\ +\ 354683070 + 35468307e \end{array}\right\} = G$$

Viz. $213489045 + 157344020e + 87239\,75ee = 274183922{,}25$

Hence $157344020e + 87239{,}75ee = 60694877{,}25$

Consequently, $180{,}3e + ee = 695{,}72 = D$

$$\text{And } \frac{D}{180{,}3 + e} = e$$

Operation	$180{,}3$)	$\dot{6}95{,}\dot{7}2$	($3{,}7 = e$
$+ e =$	$3{,}7$	549	
1. Divisor $= 183$		$146{,}72$	First $r = 10$
2. Divisor $= 184{,}0$		$128{,}80$	$+ e = 3{,}7$
		&c.	$r + e = 13{,}7 = r$ for

a second Operation, with which you may proceed, as in the last Problem, and so on to a third Operation, if Occasion require such Exactness. But this may be sufficient to shew the Method of resolving any adfected Æquation, without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last Chapter of this Part, concerning the Periphery and Area of the Circle, *&c.* wherein you will find a farther Improvement in the Numerical Solution of High Æquations than hath hitherto been publish'd.

CHAP. V.

Practical Problems *and* Rules *for finding the* Superficial Contents, *or* Area's *of Right-lin'd Figures.*

BEfore I proceed to the following Problems, it may be convenient to acquaint the Learner, that the Superficies or Area of any Figure, whether it be right-lin'd or circular, is compos'd or made up of Squares, either greater, or less, according to the different Measures by which the Dimensions of the Figure are taken or measur'd.

That is, if the Dimensions are taken in Inches, the Area will be compos'd of Square Inches; if the Dimensions are taken in Feet, the Area will be compos'd of Square Feet; if in Yards, the Area will be Square Yards; and if the Dimensions are taken by Poles or Perches, (as in Surveying of Land, *&c.*) then the Area will be Square Perches, *&c.* These Things being understood, and

and the Definitions in the 283 and 284 *Pages* well consider'd, will help to render the following Rules very easy.

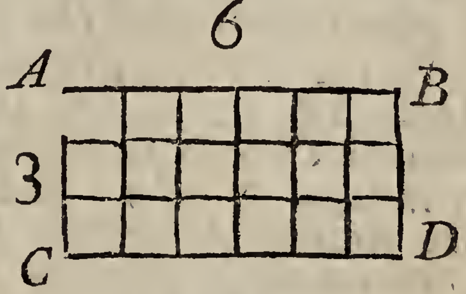
PROBLEM I.

To find the Superficial Content, or Area of a Square; *or of any right-angled* Parallelogram.

Rule. { *Multiply the Length into its Breadth, and the Product will be the Area requir'd.* (*See* Lemma 1. Page 302.)

Example, Suppose the Line $AB = 6$ Yards, and the Breadth AC or $BD = 3$ Yards.

Then $AB \times AC = 6 \times 3 = 18$ will be the Number of Square Yards contain'd in the Area of the Parallelogram $ABCD$.



This is so evident by the Figure only, that it needs no Demonstration.

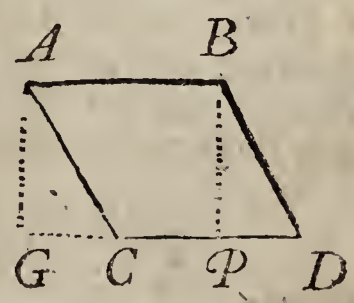
PROBLEM II.

To find the Area of any oblique-angled Parallelogram, viz. *either of a* Rhombus *or* Rhomboides.

Rule. { *Multiply the Length into its perpendicular Height, (or Breadth) and the Product will be the Area requir'd.*

That is, the Side $AB \times BP =$ the Area of the Rhombus $ABCD$. For if BP be drawn perpendicular to CD, and AG be made parallel to BP, then will $GC = PD$, and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ Rhombus $ABCD$.

But $AB \times BP = \square ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the Area of the Rhombus $ABCD$.



Example, Suppose the Side $AB = 23$ Inches, and the Perpendicular $BP = 17{,}5$ Inches, (being the shortest or nearest Distance between the two Sides AB and CD.)

Then $AB \times BP = 23 \times 17{,}5 = 402{,}5$ Square Inches, being the Area of the Rhombus requir'd.

The like may be done for any Rhomboides, whose Length and Perpendicular Breadth are given.

PROBLEM III.

To find the superficial Content, or Area of any plane Triangle.

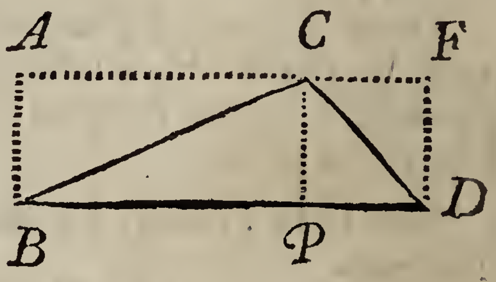
Every plane Triangle is equal to half its circumscribing Parallelogram, (41. e. 1.) which affords the following

Rule.

Multiply the Base of the given Triangle into Half its perpendicular Height, or Half the Base into the whole Perpendicular, and the Product will be the Area.

That is, $BD \times \frac{1}{2} CP$, or $\frac{1}{2} BD \times CP =$ Area of $\triangle BCD$. For $AC = BP$, $AB = CP$, and BC is common to both $\triangle\triangle$, therefore $\triangle ABC = \triangle BCP$. And for the like Reasons $\triangle CFD = \triangle CPD$.

Therefore $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABFD$. Consequently $\frac{1}{2} BD \times CP$, or $BD \times \frac{1}{2} CP$ will be the Area of the $\triangle BCD$.



Example, Suppose the Base $BD = 32$ Inches, and the perpendicular Height $CP = 14$ Inches.

Then $\frac{1}{2} BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2} CP = 32 \times 7 = 224$.

Or thus, $32 \times 14 = 448$. Then 2) 448 (224 = the Area of the Triangle BCD in Square Inches.

PROBLEM IV.

To find the Superficies, or Area of any Trapezium.

First, Divide the given Trapezium into two Triangles, by drawing a Diagonal from one of its acute Angles to the opposite Angle; and let fall two Perpendiculars (from the other two Angles) upon the Diagonal, as in the following Figure. Then

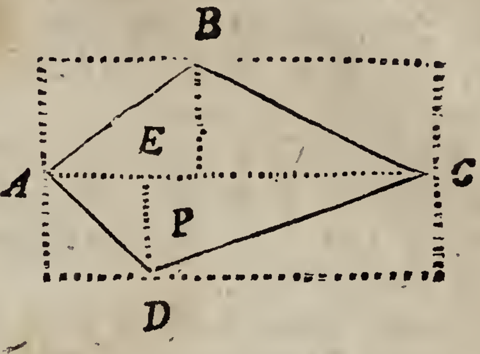
Rule. *Multiply Half the Diagonal into the Sum of the two Perpendiculars, or Half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.*

That is, $\frac{1}{2}\,\overline{AC \times BP + ED}$. Or $\overline{AC \times \frac{1}{2} BP + \frac{1}{2} ED} =$ Area of the Trapezium $ABCD$.

For the $\triangle ABC$ is Half its circumscribing Parallelogram; And the $\triangle ACD$ is also Half of its circumscribing Parallelogram, as hath been prov'd at the last Problem.

Consequently,

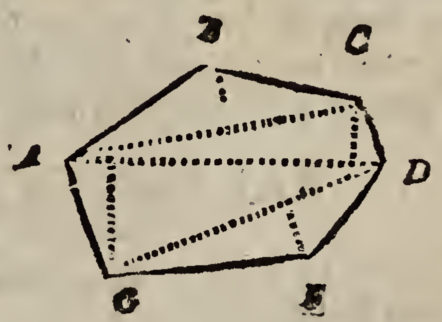
Consequently, $\overline{BP+ED}\times\frac{1}{2}AC$, or $\overline{\frac{1}{2}BP+\frac{1}{2}ED}\times AC$ will be the Area of the Trapezium, As above.



Example, Suppose the Diagonal $AC = 33$ Feet, the Perpendicular $BP = 15$ Feet, and the Perpendicular $ED = 14$ Feet. Then $BP + ED = 29$ Feet, and $\overline{BP+ED}\times\frac{1}{2}AC = 29 \times 16,5 = 478,5$. Or $AC \times \overline{\frac{1}{2}BP+\frac{1}{2}ED} = 33 \times \frac{29}{2} = 478,5$ Or thus, $29 \times 33 = 957$. Then 2) 957 (478,5 any of these Products are the Area of the Trapezium $ABCD$.

PROBLEM V.

To find the Superficial Content or Area of any irregular Polygon or many-sided Figure, which by some Authors is called a Triangulate, because (as I suppose) it must be divided into Triangles, as in the annexed Figure $ABCDFG$; by by which it is evident that the Sum of the Area's of all those Triangles, found as in the last Problem, &c. will be the Area of their circumscribing Polygon.



PROBLEM VI.

To find the Superficies, or Area of any regular Polygon, viz. of any regular Pentagon, Hexagon, Heptagon, Octagon, &c.

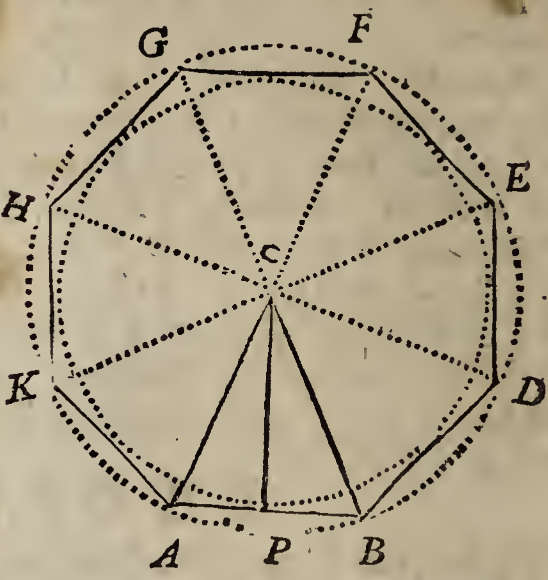
General Rule.

Multiply Half the Sum of its Sides into the Radius of the inscrib'd Circle, or Half the said Radius into the Sum of the Sides, and the Product will be the Area requir'd.

That is, $\frac{AB+BD+DE+EF+FG+GH+HK+KA}{2} : \times CP$

= the Area of the annexed Octagon; wherein it is evident, that its Area is compos'd of so many equal Isosceles Triangles as there are Number of Sides in the Polygon, *viz.* of eight Isosceles Triangles, whose Bases are the Sides of the Octagon, *viz.* $AB = BD = DE$, &c. And the Sides of those Triangles, CA, CB, CD, &c. are the Radius's of the circumscribing Circle; and their perpendicular Heights, *viz.* CP, is the Radius of the inscrib'd Circle. But

But the Area of any one of those Triangles, is $\frac{1}{2} AB \times CP$, By Problem 3. Consequently the Sum of all their Area's will be, CP into Half the Sum of all their Bases, as above.



This being equally evident in all regular Polygons whatsoever, makes the Rule general for finding their Area's.

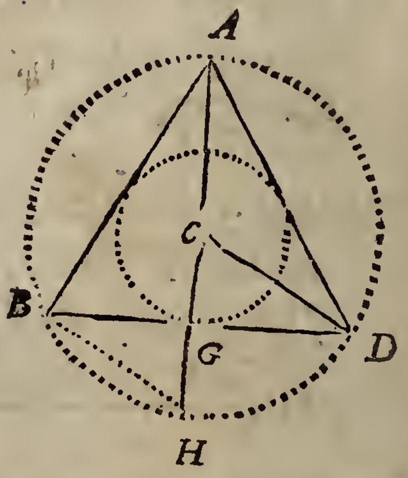
Now because it is requir'd to have the Radius of the propos'd Polygon's inscrib'd Circle, I shall here insert (and demonstrate) the Proportions that are between the Sides of several regular Polygons, and the Radius's both of their inscrib'd and circumscribing Circles; the one will help to delineate or project the Polygon, (if Occasion require it) and the other will help to find its Area.

And *First*, Of an Equilateral Triangle.

The Side of any equilateral plane Triangle, is in Proportion to the Radius of

its { Circumscribing Circle, As 1 : To 0,57735027 &c.
Inscrib'd Circle, As 1 : To 0,28867513 &c. }
And to its Perpendicular Height, As 1 : To 0,86602540 &c.

That is, $\begin{cases} AB : CD :: 1 : 0{,}57735027 \\ AB : CG :: 1 : 0{,}28867513 \end{cases}$
And $AB : AG :: 1 : 0{,}86602540$



Demonstration.

Let $AB = BD = 1$.
Then will $BG = GD = 0{,}5$
But $\square AB - \square BG = \square AG$
By *Theorem* 11.
That is, $1 - 0{,}25 = 0{,}75 = \square AG$.
Consequently, $\sqrt{0{,}75} = 0{,}86602540 = AG$:
Then $AG : AB :: AB : AH$. By *Theorem* 13.
That is, $0{,}8660254 : 1 :: 1 : 1{,}1547005 4$ &c. $= AH$.
Then $\frac{1}{2} AH = 0{,}57735027 = AC$. Again, $AG : DG :: DG : CG$.
That is, $0{,}8660254 : 0{,}5 :: 0{,}5 : 0{,}28867513 = CG$. Q. E. D.

Now, by the Help of the first of these Proportions, it will be easy to resolve the following Problem. PRO-

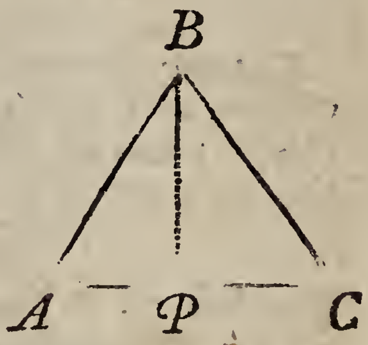
PROBLEM VII.

The Side of any equilateral plane Triangle being given, To find its Area.

Example, Suppose the Side of the propos'd Triangle ABC to be 25 Inches, *viz.* $AB = BC = CA = 25$ First $1 : 0,8660254 :: AB = 25 : 21,650635 = BP$. By *Theorem* 13.

Then AP ($= \frac{1}{2} CA$) $\times BP =$ the Area of $\triangle ABC$. By *Rule* to Problem 3.

That is, $12,5 \times 21,650635 = 270,6329$ the Area in Square Inches.



Or this Problem may be otherwise resolv'd, thus:

Let $b = AP = \frac{1}{2} AC$. Then $2b = AB$.

But $\square AB - \square AP = \square BP$. *Per* Theorem 11.

That is, $4bb - bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$

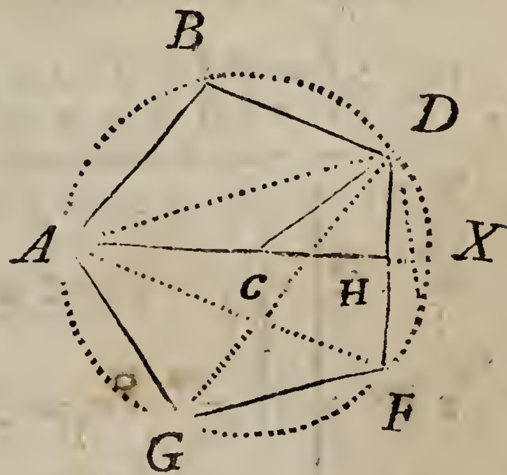
Then $b\sqrt{3bb} = BP \times \frac{1}{2} AC$. Viz. $\sqrt{3bbbb} =$ the Area of the Triangle.

Secondly, For a Pentagon.

The Side of any regular Pentagon, is in Proportion to the Radius of its { Circumscribing Circle, As 1 : To 0,85065080 &c. Inscrib'd Circle, As 1 : To 0,68819096 &c.

And to its Perpendicular Height, As 1 : To 1,53884176 &c.

Viz. $\begin{cases} AB : AC :: 1 : 0,85065080 \\ AB : CH :: 1 : 0,68819096 \\ AB : AH :: 1 : 1,53884176 \end{cases}$



Demonstration.

Let $AB = 1$. And draw the Diagonals AD, AF and DG, which will be equal to one another. Then will $AG \times DF : + AD \times GF = AF \times DG$ By *Theorem* 19.

Consequently, $AG \times DF = AF \times DG : - AD \times GF$

That is, $\square AB = \square AD : - AD \times GF = 1$

Hence it will be $AD = 1,61803398$

Then $\square AD - \square DH = \square AH$. By *Theor.* 11. But $DH = \frac{1}{2} AB$

Therefore $\sqrt{\square AD - \frac{1}{4} \square AB} = AH = 1,53884176$.

Again, $AH : AD :: AD : AX = 2\ AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.

Ergo

Ergo $\frac{\square AD}{AH} = 2\,AC = 1{,}70$[illegible]$61.$ Hence $AC = 0{,}85065080$

But $AH - AC = CH = 0{,}$[illegible]90[illegible]6 &c. Q. E. D.

From hence it will be eaſy to reſolve the following Problem.

PROBLEM VIII.

The Side of any regular Pentagon being given, To find its Area.

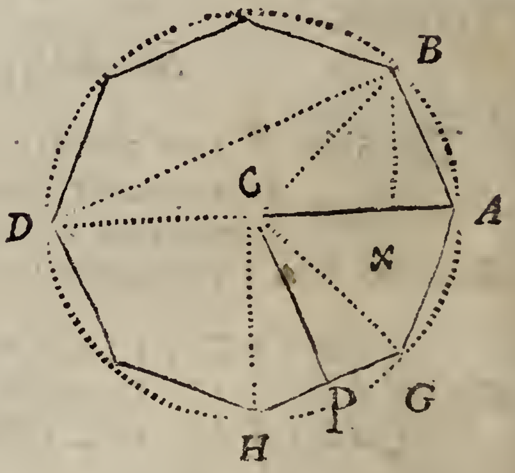
Example, Suppoſe the given Side to be 15 Inches long, then it will be, as 1 : 1,53884176 :: 15 : 22,0826264 the perpendicular Height; and by the general Rule $22{,}0826264 \times \frac{15}{2} = 165{,}619698$ the Area requir'd.

Thirdly, For an Octagon.

The Side of any regular Octagon is in Proportion to the Radius of its $\begin{cases}\text{Circumſcribing Circle,} & \text{As } 1 : \text{To } 1{,}30656296 \text{ \&c.}\\ \text{Inſcrib'd Circle,} & \text{As } 1 : \text{To } 1{,}20710678 \text{ \&c.}\end{cases}$

Viz. $\begin{cases} BA : CA :: 1 : 1{,}30656296 \\ BA : CP :: 1 : 1{,}20710678 \end{cases}$

Demonſtration.



Draw the Right-line DB, and from the Point B let fall the Perpendicular Bx upon the Diameter DA.

Then will $\triangle\, DBA$ and $\triangle\, DxB$ be alike. By *Theorem* 10 and 12.

Let $\begin{cases} b = BA = 1 \,.\; a = CA. \\ e = DB \text{ and } y = Bx. \end{cases}$

Then	1	$2a : b :: e : y$ Viz. $DA : BA :: DB : Bx$.
1 ∵	2	$\frac{2ay}{b} = e = DB$
2 ⓦ 2	3	$\frac{4aayy}{bb} = ee = \square DB$
But	4	$4aa - \frac{4aayy}{bb} = bb$
That is,		$\square DA - \square DB = \square BA$. By Theorem 11.
$4 \times bb$	5	$4bbaa - 4aayy = bbbb$
Again,	6	$\begin{cases} \frac{1}{2}aa = yy. \text{ For } Cx = Bx \\ \text{and } \square Cx + \square Bx = \square CB = aa \end{cases}$

5, 6

5,	6	7	$4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$
7 ÷	2	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
8 C	□	9	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$
9 ω	2	10	$aa - bb = \sqrt{\frac{1}{2}b^4}$
10 +	bb	11	$aa = bb + \sqrt{\frac{1}{2}b^4}$
11 ω	2	12	$a = \sqrt{: bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296$ &c. $= CA$
	Then	13	$aa - \frac{1}{4}bb = \square CP$. Viz. $\square CH - \square HP = \square CP$
13 ω	2	14	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678$ &c. $= CP$.

From hence it will be easy to find the Area of any Octagon.

PROBLEM IX.

The Side of any regular Octagon being given to find its Area.

Example, Suppose the Side given to be 12 Inches long; First, As $1 : 1,20710678 :: 12 : 14,48528136 =$ the Radius of its inscrib'd Circle. Then $12 \times 4 = 48$ is Half the Sum of its Sides; and $48 \times 14,48528136 = 695,2935$ the Area requir'd.

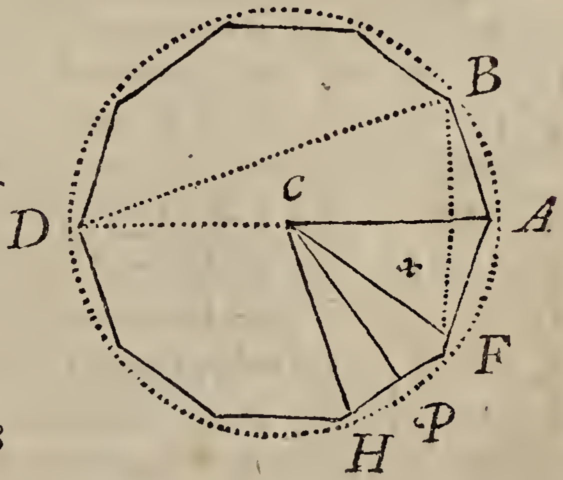
Fourthly, for a Decagon.

The Side of any regular Decagon, (viz. *a Polygon of Ten equal Sides*) *is in Proportion to the Radius of*

Its { Circumscribing Circle, As 1 : To 1,61803398 &c.
Inscrib'd Circle, As 1 : To 1,53884176 &c.

Viz. { $BA : CA :: 1 : 1,61803398$
$BA : CP :: 1 : 1,53884166$

Demonstration.



Let { $b = BA = 1$. $a = CA$
$e = DB$. and $y = Bx$

Then		1	$2a : b :: e : y$
That is			$DA . BA :: DB : Bx$
1	∵	2	$2ay = be$ and $2y = \frac{be}{a}$
	But	3	$2y : e :: 1 : 1,61803398$ See Pentagon.
3	∵	4	$\frac{1e}{1.61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$
4 ÷	1e	5	$1.61803398 = a = CA$
	Again	6	$aa - \frac{1}{4}bb = \square CP$. Viz. $\square CF - \square PF = \square CP$. By Theorem II.
That is,		7	$\sqrt{2,61803396 - 0,25} = 1,53884176 = CP$

PROBLEM X.

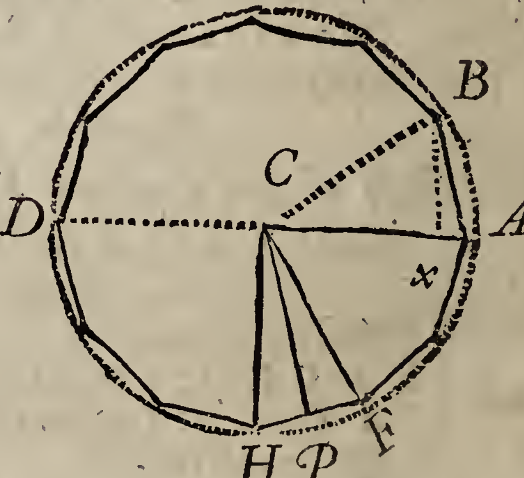
The Side of any regular Decagon being given, To find its Area.

Example, Let the given Side be 14 Inches long: Then, As $1 : 1,53884176 :: 14 : 21,543784 =$ the Radius of the inscrib'd Circle. And $14 \times 5 = 70$ is Half the Sum of its Sides. Lastly, $21,543784 \times 70 = 1508,06488$ the Area requir'd.

Fifthly, For a Dodecagon.

The Side of any regular Dodecagon, (viz. a Polygon of twelve equal Sides) is in Proportion to the Radius of its $\left\{\begin{array}{l}\text{Circumscribing Circle, As } 1 : \text{To } 1,93185165\ \&c. \\ \text{Inscrib'd Circle, As } 1 : \text{To } 1,86632012\ \&c.\end{array}\right.$

Viz. $\left\{\begin{array}{l} BA : CA :: 1 : 1,93185165 \\ BA : CP :: 1 : 1,86632012 \end{array}\right.$



Demonstration.

Let $b = BA = 1$. $a = CA$ as before
And $e = xA$. Then $a - e = Cx$

First	1	$\left\{ bb - \square Bx = ee \right.$ By *Figure.*
But	2	$Bx = \frac{1}{2}CA = \frac{1}{2}a$
2 ⊕ 2	3	$\square Bx = \frac{1}{4}aa$
1, − 3	4	$bb - \frac{1}{4}aa = ee$
4 ω 2	5	$\sqrt{bb - \frac{1}{4}aa} = e$
Again	6	$aa - \frac{1}{4}aa = aa - 2ae + ee$
Viz.		$\square CB - \square Bx = \square Cx.$
5 × 2a	7	$2a\sqrt{bb - \frac{1}{4}aa} = 2ae$
4 — 7	8	$bb - \frac{1}{4}aa - 2a\sqrt{bb - \frac{1}{4}aa} = ee - 2ae$
7, 8	9	$aa - \frac{1}{4}aa = aa + bb - \frac{1}{4}aa - 2a\sqrt{bb - \frac{1}{4}aa}$
9 +	10	$2a\sqrt{bb - \frac{1}{4}aa} = bb$
10 ⊕ 2	11	$4bbaa - aaaa = b^4$
11 +	12	$aaaa - 4bbaa = -b^4$
13 C □	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
1 ω 2	14	$aa - 2bb = \sqrt{3} = 1,7320508075$
14 + 2bb	15	$aa = 2bb + \sqrt{3} = 3.7320508075$
15 ω 2	16	$a = \sqrt{3.7320508075} = 1.93185165 = CA$
Again	17	$aa - \frac{1}{4}bb = \square CP.$ *Viz.* $\square CF - \square PF = \square CP$
17, Hence	18	$CP = \sqrt{aa - \frac{1}{4}bb} = 1\ 86632012$ Q. E. D.

Con-

CONSECTARY.

Hence, if the Side of any regular Dodecagon be given, the Radius of its inscrib'd Circle may be easily obtain'd, and thence the Area found, as in the last Problem.

The Work of the foregoing Polygons being well consider'd will help the young Geometer to raise the like Proportions for others, if his Curiosity, or Occasion requires them: And not only so, but they will also help to form a true Idea of a Circle's Periphery and Area, according to the Method which I shall lay down in the next Chapter for finding them both.

CHAP. VI.

A new and easy Method of finding the Circle's Periphery *and* Area, *to any assign'd Exactness (or Number of Figures) by one Æquation only. Also a new and facile Way of making Natural* Sines *and* Tangents.

LET us suppose (what is very easy to conceive) the Circle's Area to be compos d or made up of a vast Number of plane Isosceles Triangles, having their acutest Angles all meeting in the Circle's Center: And let us imagine the Bases of those Triangles so very small, that their Sides and their perpendicular Heights, *viz.* the Radius's of their circumscrib'd and inscrib'd Circles, (vide *Problem 6.*) may become so very near in Length to each other, as that they may be taken one for another, without any sensible Error. Then will the Peripheries of their circumscribing, and inscrib'd Circles, become (although not co-incident, yet) so very near to each other, as that either of them may be indifferently taken for one and the same Circle.

But how to find out the Sides of a Polygon (*viz.* the Bases of those Isosceles Triangles) to such a convenient Smallness, as may be necessary to determine and settle the Proportion betwixt a Circle's Diameter and its Periphery, (to any assign'd Exactness) hath hitherto been a Work which requir'd great Care and much Time in its Performance, as may easily be conceiv'd from the Nature of the Method us'd by all those who have made any considerable Progress in it, *viz.* *Archimedes, Snellius, Hugenius, Mætius, Van-Culen,* &c. These proceeded with the Bisecting of an Arch, and found the Value of its Chord to a convenient Number of

Figures at every single Bisection, repeating their Operations until they had approach'd to the Chord design'd.

And this Method is made Choice of by the learned Dr. *Wallis*, in his Treatise of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by several (very eminent in Mathematical Sciences) in order to find out some easier and more expeditious Way of approaching to the Circle's Periphery, as in *Chap.* 82, 84, 85, 86, and several other Places, he comes to this Result. (*Page* 321.)

" 'Tis true, (saith he) we might in like Manner proceed by
" continual Trisection, Quinquisection, or other Section, if we
" had for these as convenient Methods of Operation as we have
" for Bisection: But because *Euclid* shews how to bisect an Arch
" Geometrically, but not to trisect, *&c.* and the one may be done
" (Algebraically) by resolving a Quadratic Æquation; but not
" those other, without Æquations of a higher Composition, I
" therefore make Choice of a continual Bisection, *&c.*

And then he lays down these following Canons.

The Subtense of $\frac{1}{6}$	1	into 6
of $\frac{1}{12}$	$\sqrt{2-\sqrt{3}}$	into 12
of $\frac{1}{24}$	$\sqrt{2-\sqrt{2+\sqrt{3}}}$	&c. 24
of $\frac{1}{48}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$	48
of $\frac{1}{96}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	96
&c.	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}$	192
	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}$	384
	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}}}$	768
	&c.	&c.

How tedious and troublesom the Work of these complicated Extractions is, I leave to the Consideration of those, who either have had Experience therein, or out of Curiosity will give themselves the Trouble of making Trial.

Again, in *Page* 347, the Doctor inserts a particular Method propos'd by *Leibnitius*, publish'd in the *Acta Eruditorum* at *Leipsick*, for the Month of *February* 1682, in order to find the Circle's Area, and consequently its Periphery, which is this:

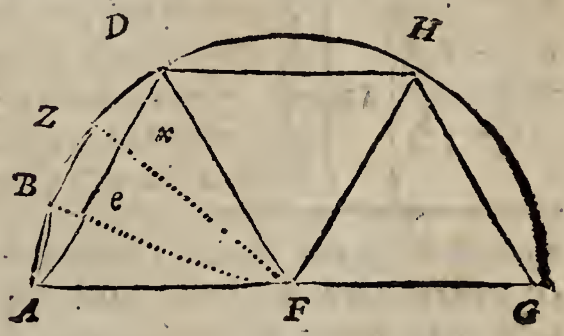
As 1 : To $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$ *&c.* Infinitely : : So is the Square of the Diameter : to the Circle's Area. But this convergeth so very slowly, that it is not worth the Time to pursue it.

I shall here propose a new Method of my own, whereby the Circle's Periphery (and consequently its Area) may be obtain'd infinitely

infinitely near the Truth, with much greater Eaſe and Expedition, than either that of Biſection, or that of *Leibnitius*, as above, or any other Method that I have yet ſeen, it being perform'd by reſolving only one Æquation, deduc'd by an eaſy Proceſs from the Property of a Circle, (known to every Cooper) which is this.

The Radius of every Circle is equal to the Chord of one ſixth Part of its Periphery.

That is, $AD = DH = HG$, the Chords of $\frac{1}{3}$ Part of the Semicircle, are each equal to AF its Radius.



Then if the Arch AD be triſected, it will be $AB = BZ = ZD$.

Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB \end{cases}$ Quere a.

Then	1	$R : a :: a \frac{aa}{R} = Be$
And	2	$R : a :: R : -\frac{aa}{R} : c - 2a$
That is,	3	$FB : BZ :: Fe : ex = AD - 2a$
For		$\triangle AFB$ and $\triangle BAe$, are alike. And $AB = Ae = Dx$, &c.
2 ∵	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 × &c.	5	$3Ra - aaa = RRc$. That is, $3a - aaa = 1$. Here $a =$ the Chord of $\frac{1}{18}$ Part of the Circle. For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$.

Next, *To triſect the Arch AB.*

Let	1	$3y - y^3 = a$ the laſt Chord.
1 ⓖ 3	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
1 × 3	3	$9y - 3y^3 = 3a$
3 — 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$. Here $y =$ the Chord of $\frac{1}{54}$ Part of the Circle.

Again, *To triſect the Arch, whereof y is the Chord.*

Let	1	$3a - a^3 = y$
1 ⓖ 3	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
1 ⓖ 5	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$

1 ⊕ 7	4	$2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + 945a^{15} = y^7$
1 ⊕ 9	5	$19683a^9 - 59049a^{11} + 78732a^{13} - 61236a^{15} = y^9$
1 × 9	6	$27a - 9a^3 = 9y$
2 × 30	7	$810a^3 - 810a^5 + 270a^7 - 30a^9 = 30y^3$
3 × 27	8	$6561a^5 - 10935a^7 + 7290a^9 - 2430a^{11} + 405a^{13} - 27a^{15} = 27y^5$
4 × 9	9	$19683a^7 - 45927a^9 + 45927a^{11} - 25515a^{13} + 8505a^{15} = 9y^7$
6 − 7 + 8 − 9 + 5	10	$\left.\begin{array}{l} 27a - 819a^3 + 7371a^5 - 30888a^7 \\ + 72930a^9 - 107406a^{11} \\ + 104652a^{13} - 69768a^{15} \end{array}\right\} = 1$
		Here $a =$ the Chord of $\frac{1}{162}$ Part of the Circle.

Proceeding on in this Method of continually trisecting the Arch of every new Chord, and still connecting the produc'd Æquations into one, as in the two last Trisections, it will not be difficult to obtain the Chord of any assign'd Arch, how small soever it be.

Now in order to facilitate the Work of raising these Æquations to any considerable Height, it will be convenient to add some few useful Observations concerning the Nature, and of such Contractions as may be safely made in them; which, being well understood will render the Work very easy.

1. I have observ'd that every Trisection will gain or advance one Figure in the Circle's Periphery, but no more. Therefore so many Places of Figures as are at first design'd to be perfect in the Periphery, so many Trisections must be repeated to raise an Æquation, that will produce a Chord answerable to that Design.

2. I have also found, that all the superior Powers (of a) whose Indices are greater than the Number of Trisections, *viz.* whose Indices are greater than the Number of design'd Figures, may be wholly rejected as insignificant.

3. When once the Number of Trisections, and thence the highest Power (of a) is determin'd, the third Process, *viz.* the third Trisection) may be made a fix'd or a constant Canon; for by it, and Multiplication only, all the succeeding Trisections, how many soever they are) may be completed, without repeating the several Involutions.

4. In raising and collecting the Co-efficients of the several Powers, (of a) it will be sufficient to retain only so many significant Figures

Figures (at a^3) as there is deſign'd to be Places of Figures in the Periphery; (or at moſt but two more) and every ſucceeding ſuperior Power may be allow'd to decreaſe two Places of ſignificant Figures. But herein great Care muſt be taken to ſupply the Places of thoſe Figures that are omitted with Cyphers, that ſo the whole and exact Number of Places may be truly adjuſted; otherwiſe all the Work will be erroneous.

Now the Number of thoſe ſupplying Cyphers may be very conveniently denoted by Figures plac'd within a Parentheſis; thus, $576(8)\,a^3$, may ſignify $5760000000\,a^3$, as in the following Æquations. The like may be done with Decimal Parts, thus (,7) 658 may ſignify ,0000000658, &c. which will be found very uſeful in the Solution of theſe, and the like Æquations.

The aforeſaid Contractions may be ſafely made, becauſe both the ſuperior Powers of a, which are rejected; as alſo thoſe Numbers that are omitted in the Co-efficients (and ſupply'd with Cyphers) would produce Figures ſo very remote from Unity, as that they would not affect the Chord deſign'd. That is, they would not affect the Chord in that Place wherein the deſign'd Periphery is concern'd, as will in Part appear in the following Example.

If theſe Directions be carefully minded, it will be eaſy to raiſe an Æquation that will produce the Side of a regular Polygon, whoſe Number of Sides ſhall be vaſtly numerous, conſequently infinitely ſmall. But I preſume it will be ſufficient for an Example, to find the Side of a Polygon conſiſting of 258280326 equal Sides; that is, if I find the Chord of $\frac{1}{258280326}$ Part of the Circle's Periphery, and that requires but ſixteen Triſections, which being order'd as before directed, will produce this Æquation.

$$\left\{\begin{array}{l} 43046721a - 3323601794869686 12(4)a^3 \\ +7698376531997 14(20)a^5 - 84912185328 41(35)a^7 \\ +54633331143(50)a^9 - 230083348(66)a^{11} \\ +6830988(79)a^{13} - 15072(94)a^{15} \end{array}\right\} = 1$$

Here the Value of a will have 23 Places of Figures true; that is, the Sides of the inſcrib'd and circumſcrib'd Polygons, will be exactly the ſame to 23 Places of Decimal Parts, but not farther; all which may be eaſily obtain'd at two Operations. And for the firſt, will be ſufficient to take only three Terms of the Æquation; which will admit of being yet farther contracted, thus

$$\text{Let} \left\{\begin{array}{l} 43046721a - 3323601794(12)a^3 \\ +76983765(27)a^5 \end{array}\right\} = 1$$

And

And let $r+e=a$. Then rejecting all the Powers of e, that arise by Involution above eee,

it will be $r^3+3rre+3ree+eee=aaa$

And $r^5+5r^4e-10r^3ee+10rreee=a^5$

Then the first single Value of r may be thus found;

$43046721)\ 1{,}00000000\ ({,}00000002=r$

This $,00000002=r$ being duly involv'd, and its Powers multiply'd into their respective Co-efficients, will produce

$$\left.\begin{array}{l} +{,}86093442+43046721e \\ -{,}02658881-\ 3988322e-199416(9)ee-3324(18)eee \\ +{,}00024635+\ \ 61587e+\ 6159(9)ee+\ 308(18)eee \end{array}\right\}=1$$

Viz. $83459196+39119986e-193257(9)ee-3016(18)eee=1$

Hence $39119986e-193257(9)ee-3016(18)eee=0{,}16540804$

All the Terms of this last Æquation being divided by 193257(9 the Co-efficient of ee, it will then become

$,0000002024e-ee-156(5)eee=,00000000000000008558968=D$

Consequently, $\left\{\dfrac{D+156(5)\,eee}{,0000002024-e}\right.=e$

Operation.

$,0000002024)\ ,00000000000000008558968\ (,000000004=e$

$-=,0000000043:+,000000000000000000009984=156(5)eee$

1 Di.,000000198)	,00000000000000008568952	(000000004327
2 Di.,0000001981	792	
	6489	
	5943	
	5465	
First $r=,00000002$	3962	
$+e=,000000004327$	&c.	

$r+e=,000000024327=a$. Or rather new r for a 2d Operation.

Now if this first Value of $a=.000000024327$ were not continued to more Places of Figures, by a second Operation, but only multiply'd into the Number of Chords,

Viz. $,000000024327\times258280326=6{,}28318539$ &c. the Periphery of that Circle, whose Diameter is 2, nearer than either *Archimedes*

Archimedes, or *Mœtius*'s Proportion: For *Archimedes* makes it 6,285714 &c. *viz.* As 7 to 22. And *Mœtius* makes it 6,28318584 &c. *viz.* as 113 to 355.

But if the whole Æquation before propos'd be now taken, and we proceed to a ſecond Operation, the Value of a may be increas'd with twelve Places of Figures more, and thoſe may be obtain'd by plain Diviſion only.

Thus, let $r+e=a$, as before, and let all the Powers of e be now rejected as inſignificant;

$$\text{Then will }\left\{\begin{array}{l} r+e=a \\ r^3+3r^2e=a^3 \\ r^5+5r^4e=a^5 \\ r^7+7r^6e=a^7 \end{array}\right\}\text{ and }\left\{\begin{array}{l} r^9+9r^8e=a^9 \\ r^{11}+11r^{10}e=a^{11} \\ r^{13}+13r^{12}e=a^{13} \\ r^{15}+15r^{14}e=a^{15} \end{array}\right.$$

The ſeveral Powers of $r=,000000024327$ being rais'd and multiply'd into their reſpective Co-efficients, will produce theſe following Numbers.

$$\left.\begin{array}{lr} +1,04719758 1767 & +\ 43046721e \\ -\ ,047849196598394865- & 5900751e \\ +\ ,000655906484595355+ & 134810e \\ -\ ,000004281440413375- & 1232e \\ +\ ,000000016302517863+ & 6e \\ -\ ,000000000040631167- & 0e \\ +\ ,000000000000071388+ & 0e \\ -\ ,000000000000000093- & 0e \end{array}\right\}=1$$

$$Viz.\ 1,000000026474745106+37279554e=1$$

$$\text{Hence } 37279554e=-,000000026474745126=D$$

$$\text{Or rather}-37279554e=,000000026474745106=D$$

$$\text{Conſequently, } \frac{D}{37279554}=-e$$

Operation.

$$37279554)\ ,000000026474745106(\ (0^{15})710167967=-e$$

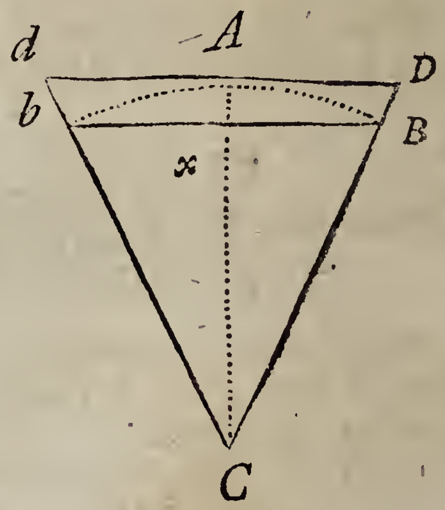
$$\begin{array}{r} 260956878 \\ \hline 37905730 \\ 37279554 \\ \hline 62617660 \\ 37279554 \\ \hline \text{\&c.} \end{array}$$

Laſt

Laſt $r = ,000000024327$
$- \; e = .00000000000000000710167967$

$r - e = .00000002432699928983 2033 = a$ the Chord or Side of the Polygon requir'd.

The next Work will be to examine how many Places of theſe Figures will hold true to the Circle's Periphery: In order to that let a be repreſented by the Chord $B\,b$, in the annexed Scheme; and let $Bx = xb$. Then will $Bx = \frac{1}{2}\,a = (,7)\,121634996440160165$ And $\square\,BC - \square\,Bx = \square\,Cx$. Let the Radius BC 1 as before. Then will $\surd\,\square\,BC - \square\,Bx = Cx = ,9999999999999999$, &c.



But $Cx : xB :: CA : AD$ } Per Fig.
Or $Cx : Bb :: CA : Dd$ }
Ergo $Dd = (,7)24326999289832 0354$ the Side of the circumſcribing Polygon.
Then will $a \times 258280326$ be the Perimeter of the inſcrib'd Polygon.

And $Dd \times 258280326$ will be the Perimeter of the circumſcribing Polygon.

That is, 6,2831853071795859 = the Perimeter of the inſcrib'd Polygon.

And. 6,2831853071795865 = the Perimeter of the circumſcrib'd Polygon.

Hence 'tis evident, that the Circle's Periphery, whoſe Diameter is 2, may be concluded 6,2831853071795864 true, becauſe the Perimeters of the inſcrib'd and circumſcrib'd Polygons are ſo far very near being co-incident, or the ſame.

'Tis poſſible there may be ſome who will think this is tedious and troubleſom Work; but if thoſe pleaſe to conſider, that if this Periphery were to be found by the aforeſaid Method of Biſection, it would require theſe following Extractions.

Viz. $\surd:2 - \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2$
$+ \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2$
$+ \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:2$
$+ \surd:2 + \surd\;2 + \surd:2 + \surd:2 + \surd:2 + \surd:2 + \surd:3$ Multiply'd into 402809984.

Here the firſt Root (*viz.* $\surd$ 3.) muſt be extracted at leaſt to one hundred and two Places of Figures. The ſecond Root (*viz.*

$\surd:2$

$\sqrt{:2+\sqrt{:3}}$) must have 99 Places of Figures in it. The third Root (*viz.* $\sqrt{:2+\sqrt{:2+\sqrt{3}}}$) must have 96 Places in it, *&c.* every Extraction being allow'd to decrease three Places, that so the last Root (*viz.* the Chord sought) may consist of 24 Places of Figures, as above.

I say, whoever duly considers the Trouble of these so often repeated Extractions, will, I presume, be pleas'd with what I have done. For truly when I consider of the great Time and Care requir'd in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that Way until he had found the Circle's Periphery to thirty six Places of Figures, to wit, 6,28318530717958647692528676655900576. These Numbers are said to be engraven upon his Tomb-stone in St. *Peter*'s Church in *Leyden*, for a Memorial of so great a Work.

Having thus obtain'd the Circle's Periphery, its Area may easily be found (to the same Number of Figures) by *Problem 6*.

That is, if Half the Periphery of any Circle be multiply'd into Half its Diameter, the Product will be that Circle's Area, as will appear farther on. Therefore 3,14159265358979 3 will be the Area of the Circle whose Diameter is 2.

Thus I have shew'd the young Geometer how to find the Circle's Periphery and Area to what Exactness he pleases to approach; for precisely true they cannot be found, notwithstanding the late Pretensions of a certain *Frenchman*, who hath publish'd to the World, (in the *Works of the Learned*) that after twenty five Years Study he had found the Quadrature of the Circle: But if he had perus'd the 83d Chapter of Dr. *Wallis*'s Algebra, he might there have seen his Error, *viz.* the Impossibility of what he pretended to; for it is as impossible to square the Circle (that is, to find its true Area) as it is to find the Root of a Surd Number.

Note, What I have here propos'd and done by the Trisection of an Arch, may as easily and much more speedily be perform'd by Quinquesection or Septisection, *&c.* But because the Scheme for Trisection is more simple and may be easier understood by a Learner than those of the other Sections, (of which see my *Compendium of* Algebra. *Pages* 76 and 79) I have for that Reason made Choice of Trisection.

As to the Proportion of one Circle to another, and of the Circle to the Ellipsis, *&c.* those shall be fully shew'd when we come to the fifth Part.

Before I conclude this Part, I ſhall make ſome Uſe or Application of the above found Periphery, in finding the Quantity of Angles, which is done by the Help of Right-lines, call'd Sines and Tangents, the Length whereof is calculated to every Degree and Minute of a Quadrant, by much Labour. But I ſhall here ſhew how to find the Natural Sine (and conſequently the natural Tangent) of any propos'd Arch or Angle, by two Æquations, without the Help of any preceding Sine as uſual; which I did ſome Years ago communicate to the ingenious Mr. *Joſeph Raphſon*, and he ſo well approv'd of them, as to make them the 20th and 21ſt Problems in the ſecond Edition of his *Analyſis Æquationum Univerſalis*.

And becauſe in finding the Quantity of Angles, every Circle is ſuppos'd to be divided into 360 equal Parts, call'd Degrees; every Degree is ſubdivided into 60 Parts, call'd Minutes; and every Minute into 60 Seconds, &c. (See *Page* 294.)

Therefore 360) 6,2831853 &c. (0,0174532925 &c, is an Arch of the above-found Periphery, equal to the Arch of one Degree.

And 60) 0,0174532925 &c. (0,0002908882 &c. = the Arch of one Minute.

Then if the given Arch (or Angle) be leſs than 45 Degrees, reduce it into Minutes, and multiply thoſe Minutes into this conſtant Multiplicator, *viz.* 0,0002908882 calling the Product p. And for the Sine ſought put a. Then it will be

$$-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp.$$

Example.

Let it be requir'd to find the Sine of 19°. 13′ = 1153′

Here 0,0002908882 × 1153 = 0.3353940946 = p

And $-a^4 + 4{,}024729a^3 - 199{,}049611aa + 80{,}494583a =$ 5,06201394.

Let $r + e = a$

$$\text{Then} \begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + 3ree = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$$

Note, In this Caſe the firſt r may always be taken equal to the firſt Figure in the Product $= p$. *Viz.* here $r = 0{,}3$ which being involv'd as its Powers direct, and thoſe Powers multiply'd into the reſpective Co-efficients of the Æquation, it will be

$$\left\{\begin{array}{llll} +\ 24{,}1483 & +\ 80{,}49e & & \\ -\ 17{,}9144 & -\ 119{,}43e & -\ 199{,}05ee & \\ +\ 0{,}1086 & +\ 1{,}08e & +\ 3{,}62ee & \\ -\ 0{,}0081 & -\ 0{,}11e & -\ 0{,}54ee & \end{array}\right\} = 5{,}06201394$$

Viz. $6{,}3344 - 37{,}97e - 195{,}97ee = 5{,}06201$

Hence

Hence $37,97e + 195,97ee = 1,27239$

And $0,193e + ee = 0,006492 = D$

Theorem. $\left\{ \dfrac{D}{,193 + e} = e \right.$

Operation. $,193$) $0,006492$ ($0,029 = e$

$+ e = ,029$) 42

1. Divisor ,21   2292

2. Divisor ,222   1998

First $r = 0,3$

$+ e = 0\,029$

$r + e = 0,329 = r$ for a second Operation.

Which being involv'd and multiply'd, &c. as before, will produce these Numbers.

$+ 26,48271781 + 80,49458e$

$- 21,54532894 - 130,97464e - 199,0496ee$

$+ 0,14332578 + 1,30692e + 3,9724ee$

$- 0,01171611 - 0,14244e - 0,6494ee$

Viz. $5,06899854 - 49,31558e - 195,7266ee - 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$; which being divided by $195,7266$ the Co-efficient of ee will become

$,25196e + ee = ,0000356854 = D$

Then $\left\{ \dfrac{D}{,25196 + e} = e \right.$

Operation. $0,25196$) $,0000356854$ ($0,0001415 = e$

$+ e = 0,00014$ 2520

1. Divisor 0,2520   104854

2. Divisor 0,25210   100840

40140

25210

Last $r = 0,329$ &c.

$+ e = 0,0001415$

$r + e = a = 0,3291415$ being the natural Sine of $19° . 13'$ As was requir'd.

Thus you may find the right Sine of any Arch or Angle less than 45 Degrees.

But

But if the given Arch be greater than 45 Degrees, you muſt take its Compliment to 90°. *viz.* ſubtract it from 90 Degrees, and reduce the Remainder into Minutes, As before.

Then multiply the Square of thoſe Minutes into this conſtant Multiplicator, 0,000000084616 calling their Product p, and putting $a =$ the Sine ſought, As before. Then will

$$a^4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$$

Example.

Suppoſe it were requir'd to find the Sine of 75°. 32′. Or (which is the ſame thing) to find the Co-ſine of 14°. 28′. = 868′ whoſe Square 753424 × 0,000000084616 = 0,0637517251 8 = p. Hence the Æquation in Numbers will be

$$aaaa + 28aaa + 197{,}295062aa - 21{,}114814a = 190{,}8361102588$$

Let $r - e = a$ And $r = 1$

$$\text{Then} \begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note. I here take $r = 1$ becauſe the Arch is ſo near to 90°. and therefore I make it $r - e = a$.

$$\text{Then} \left.\begin{cases} - 21{,}1148 + 21{,}11e \\ + 197{,}2956 - 394{,}59e + 197{,}29ee \\ + 28{,}0000 - 84{,}00e + 84{,}00ee \\ + 1{,}0000 - 4{,}00e + 6{,}00ee \end{cases}\right\} = 190{,}8361$$

Viz. $205{,}1808 - 461{,}48e + 287{,}29ee = 190{,}8361$

Hence $461{,}48e - 287{,}29ee = 14{,}3447$

And $1{,}606e - ee = {,}049930 = D$

Theorem. $\left\{ \dfrac{D}{1{,}606 - e} = e \right.$

Operation. $1{,}606$) $\dot{,}0\dot{4}99\dot{3}0$ ($0{,}031 = e$

	$-e = 0{,}031$	471	
1. Diviſor	1,57	2830	Firſt $r = 1{,}000$
2. Diviſor	1,575	1575	$-e = 0{,}031$
		&c.	$r - e = 0{,}969 = r$

for a Second Operation; which being involv'd as before, will produce theſe following Numbers.

— 20

$$\begin{array}{llll}
- 20,460254766 & + 21,11481e & & \\
+ 185,252368710 & - 382.35783e & + 197,2951ee & \\
+ 25,475889852 & - 78,87272e & + 81,5960ee & \\
+ 0,881647759 & - 3,63941e & + 5,6337ee &
\end{array}$$

Viz. $191,149651515 - 443,75515e + 284,5248ee =$ 190836110159

Hence it will be $443\,75515e - 284,5248ee = 0,313541256$

And $1,55963e - ee = ,0011019821 = D$

Then $\left\{ \dfrac{D}{1,55963 - e} = e \right.$

Operation. $1,55963$) $0,0011009821$ ($0,0007068 = e$

$- e = 0,00070$ 109123

1. Divisor $1\,5589$ 1075210

2. Divisor $1,55893$ 935358

1398520

Last $r = 0,969$ 1247144 &c.

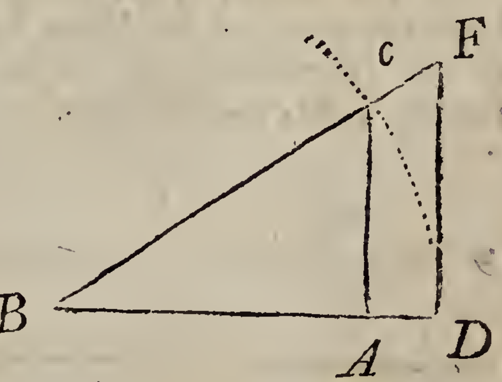
$- e = 0,0007068$

$r - e = a = 0,9682932$ the Sine of $75^\circ.\ 32'$. As was requir'd.

Having found the Sine and Co-sine of any Arch, the Tangent is usually found by this Proportion;

Viz. { *As the Co-sine of any Arch: is to the Sine of that Arch:: so is the Radius: to the Tangent of the same Arch.*

For supposing $BC = BD$ Radius, AC the Sine of the Arch CD. Then BA is the Co-sine, and FD the Tangent of the same Arch. But $BA : CA :: BD : FD$, &c.



Now by this Proportion, there is requir'd to be given, both the Sine and Co-sine of the same Arch, to find the Tangent.

'Tis true, if the Radius, and either the Sine or the Co-sine be given, the other may be found

Thus, $\sqrt{\square BC - \square CA} = BA$. Or $\sqrt{\square BC - \square BA} = CA$.

But if either the Sine or Co-sine be given, the Tangent may (I presume) be more easily found by the following Theorems.

Let

Let $BC = 1$. $CA = s$. $BA = x$. and $FD = T$. Then if s be given, T may be found by this

$$\text{Theorem} \left\{ \sqrt{\frac{ss}{1 - ss}} = T \right.$$

Or if x be given, T may be found by this

$$\text{Theorem} \left\{ \sqrt{\frac{xx}{1 - xx}} = T \right.$$

Let the Sine of 19°. 13′. (before found, be given, *viz.* $0{,}3291415 = s$. To find T the Tangent of the ſame Arch.

First, $0{,}3291415 \times {,}3291415 = 0{,}108334127 = ss$.

Again $1 - 0{,}108334127 = 0{,}891665873 = 1 - ss$.

Then $0{,}891665873)\,0{,}108334127\,(0{,}1214963253$

And $\sqrt{0{,}1214963253} = 0{,}3485632 = T$ the Tangent of 19°. 13′. As was requir'd.

And ſo you may proceed to find $T =$ the Tangent, when $x =$ the Co-ſine is given.

Perhaps it may be here expected, that I ſhould have ſhew'd and demonſtrated (or at leaſt have inſerted) the Proportions from whence the foregoing Æquations for making Sines were produc'd; but I have omitted them, as alſo their Uſe in computing the Sides and Angles of plane Triangles by the Pen only, (*viz.* without the Help of Tables) for the Subject of another Diſcourſe hereafter, if Health and Time permit.

In the mean time, what is here done may ſuffice to ſhew, that the making of Sines by ſuch a laborious and operoſe Way, as was formerly uſed, is in a great Meaſure overcome, which, I think, I may juſtly claim as my own.

AN INTRODUCTION TO THE Mathematicks.

PART IV.

Thomas CHAP. I. Dimin

Definitions of a Cone and its Sections.

THere are ſeveral Definitions given of a Cone: The learned Dr. *Barrow* upon *Euclid* hath it thus:

" A Cone (*ſaith he*) is a Figure made when one Side " of a Rectangled Triangle, (*viz.* one of thoſe Sides " that contain the Right-angle) remaining fix'd, the " Triangle is turn'd round about, till it return to the " Place from whence it firſt mov'd. And if the fix'd " Right-line be equal to the other, which containeth the " Right-angle, then the Cone is a rectangled Cone; " but if it be leſs, it is an obtuſe-angled Cone; if great- " er, an acute-angled Cone. The Axis of a Cone, is " that fix'd Line about which the Triangle is mov'd. " The Baſe of a Cone is the Circle, which is deſcrib'd " by the Right-line mov'd about, (Defin. 18, 19, 20. " *Euclid* 11.

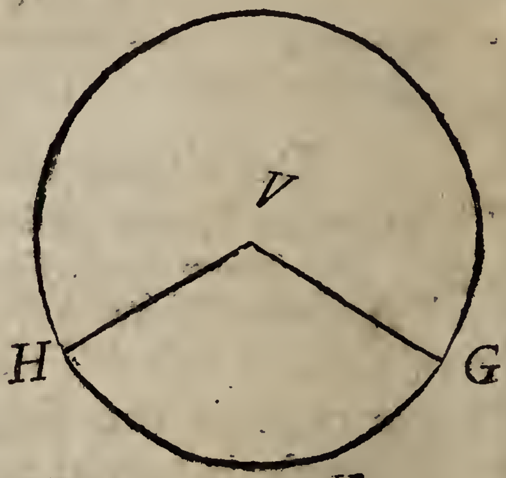
Sir *Jonas Moor*, in his *Treatiſe of Conical Sections*, (taken out of the Works of *Mydorgius*) defines it thus:

" If a Line of ſuch a Length as ſhall be needful, ſhall " upon a Point fix'd above the Plane of a Circle, ſo " move about the Circle, until it return to the Point " from whence the Motion begun, the Superficies that " is made by ſuch a Line, is call'd a conical Superficies,

" and the ſolid Figure contain'd within that Superficies
" and the Circle, is call'd a Cone. The Point remain-
" ing ſtill is the Vertex of the Cone, &c.

Altho' both theſe Definitions are equally true, and, with a little Conſideration, may be pretty eaſily underſtood; yet I ſhall here propoſe one, very different from either of them, and, as I preſume, more plain and intelligible, eſpecially to a Learner.

If a Circle deſcrib'd upon ſtiff Paper (or any other pliable Matter) of what Bigneſs you pleaſe, be cut into two, three, or more Sectors, either equal, or unequal, and one of thoſe Sectors be ſo roll'd up as that the Radius's may exactly meet each other, it will form a Conical Superficies.



That is, if the Sector *HVG* be cut out of the Circle, and ſo roll'd up as that the Radius's *VH* and *VG* may juſt meet each other in all their Parts, it will form a Cone, and the Center *V* will become a ſolid Point, call'd the *Vertex* of the Cone; the Radius *VH* being every where equal, will be the Side of the Cone, and the Arch *HG* will become a Circle, whoſe Area is call'd the Cone's Baſe.

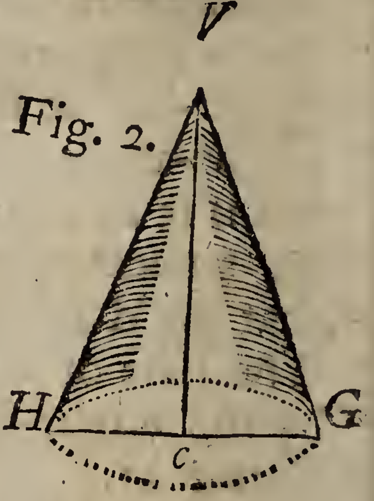
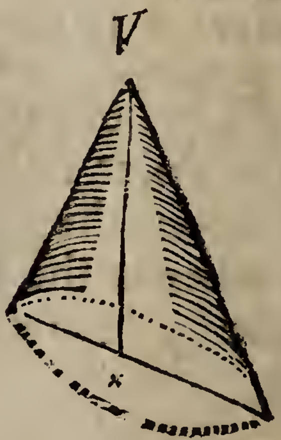


Fig. 2.

A Right-line being ſuppos'd to paſs from the Vertex or Point *V*, to the Center of the Cone's Baſe, as at *C*, that Line (*viz. VC*) will be the *Axis*, or perpendicular Height of the Cone.

If a Solid be exactly made in ſuch a Form, it will be a complete or perfect Cone; which I ſhall all along call a right Cone, becauſe its Axis *VC* ſtands at Right-Angles with the Plane of its Baſe *HG*, and its Sides are every-where equal.



Any Cone whoſe Axis is not at Right-Angles with the Plane of its Baſe, may be properly call'd an imperfect Cone, becauſe its Sides are not every where equal (as in the annexed Figure.) Now, ſuch an imperfect Cone is uſually call'd a Scalene, or oblique Cone.

Any

Any Solid Cone may be cut by Planes, (which I shall all along hereafter call Right-lines) into five *Sections*.

SECT. I.

If a Right-Cone be cut directly through its Axis, the Plane or Superficies of that Section will be a plane Isosceles Triangle, as *HVG* Figure 2. *viz*. the Sides (*HV* and *VG*) of the Cone will be the Sides of the Triangle, the Diameter (*HG*) of the Cone's Base will be the Base of the Triangle, and (*VC*) its Axis will be the perpendicular Height of the Triangle.

SECT. II.

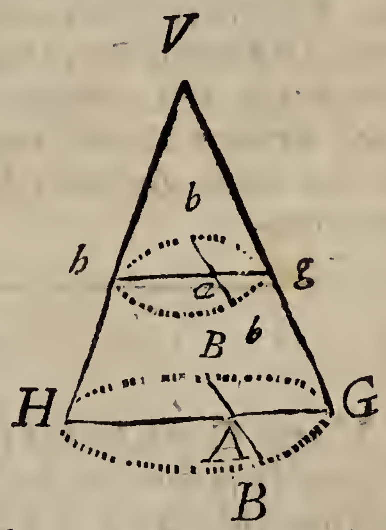
If a Right-Cone be cut (any where) off by a Right-line parallel to its Base, as *h g* (it will be easy to conceive, that) the Plane of that Section will be a Circle, because the Cone's Base is such; wherein one thing ought to be clearly understood, which may be laid down as a *Lemma* to demonstrate the Properties of the following Sections.

Lemma.

If any two Right-lines inscrib'd within a Circle, do cut or cross each other (as *h g* doth *b b* in the annexed Figure) the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line, (See *Theorem* 15, *Page* 315.)

That is, $ha \times ga = ba \times ab$ } &c.
And $HA \times GA = BA \times AB$ }

Consequently, if $ba = ab$
And if $BA = AB$.
Then it will be, $ha \times ga = \square\, ba$.
And in the Cone's Base $HA \times GA = \square\, BA$.



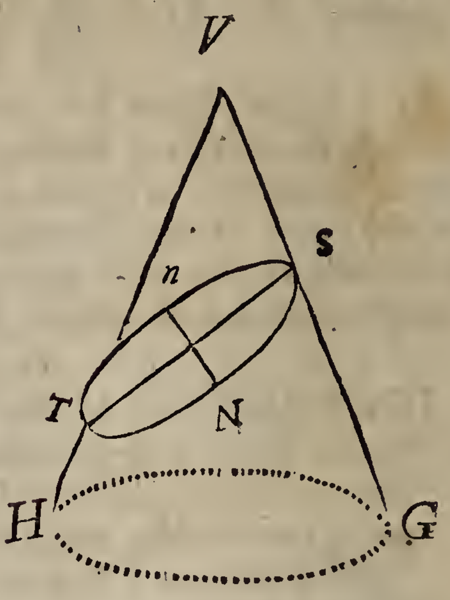
SECT. III.

If a Right Cone be (any where) cut off by a Right-line, that cuts both its Sides, but not parallel to its Base (as *TS* in the following Figure) the Plane of that Section will be an Ellipsis (vulgarly call'd an Oval, *viz*. an oblong or imperfect Circle, which hath several Diameters, and two particular Centers. That is,

1. Any Right-line that divides an Ellipsis into two equal Parts, is call'd a Diameter; amongst which, the longest and the shortest are particularly distinguish'd from the rest, as being of most general Use; the other are only applicable to particular Cases.

 2. The

2. The longest Diameter (as *TS*) is called the Transverse-Diameter, or Transverse-Axis; being that Right line which is drawn through the Middle of the Ellipsis, and doth shew or limit its Length.



3. The shortest Diameter, call'd the Conjugate-Diameter, is a Right-line that doth intersect or cross the Transverse-Diameter at Right-angles, in the middle or common Center of the Ellipsis, (as *Nn*) and doth limit the Ellipsis Breadth.

4. The two Points, which I call particular Centers of an Ellipsis, (for a Reason which shall be shew'd farther on) are two Points in the Transverse-Diameter, at an equal Distance each Way from the Conjugate-Diameter, and are usually call'd *Nodes*, *Focus*'s, or burning Points.

5. All Right-lines within the Ellipsis that are parallel to one another, and can be divided into two equal Parts, are call'd *Ordinates* with respect to that Diameter which divides them. And if they are parallel to the Conjugate, *viz.* at Right-angles with the Transverse-Diameter, then they are call'd *Ordinates rightly apply'd*. And those two that pass through the Focus's, are remarkable above the rest, which being equal and situated alike, are call'd both by one Name, *viz. Latus Rectum* or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

SECT. IV.

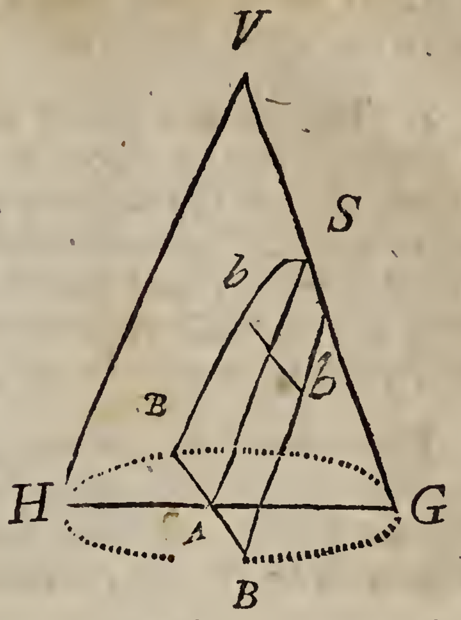
If any Cone be cut into Two Parts, by a Right-line parallel to one of its Sides, (as *S A* in the following Scheme) the Plane of that Section (*viz. S b B A B b S*) is call'd a *Parabola*.

1. A Right-line being drawn through the Middle of any Parabola, (as *S A*) is call'd its Axis or intercepted Diameter.

2. All Right lines that intersect or cut the Axis at Right-angles, (as *B B*, and *b b* are suppos'd to cut or cross *S A*) are call'd Ordinates rightly apply'd, (as in the Ellipsis) and the greatest Ordinate as *B B*, which limits the Length of the Parabola's Axis (*S A*) is usually call'd the Base of the Parabola.

3. That

3. That Ordinate which paſſes through the Focus, or Burning-Point of the Parabola, is call'd the *Latus Rectum*, or Right-Parameter (as in the Ellipſis) becauſe by it all the other Ordinates are proportion'd, and may be found.

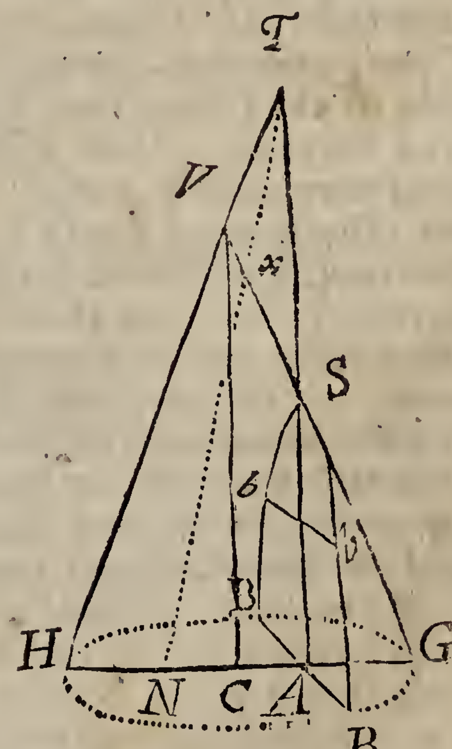


4. The Node, Focus, or Burning-Point of the Parabola, is a Point in its Axis (but not a Center as in the Ellipſis) diſtant from the Vertex or Top of the Section, (*viz.* from *S*) juſt one fourth Part of the *Latus Rectum*; as ſhall be ſhew'd farther on.

5. All Right-lines drawn within a Parabola parallel to its Axis are call'd Diameters; and every Right-line that any of thoſe Diameters doth biſect or cut into two equal Parts, is ſaid to be an Ordinate to that Diameter which biſects it.

SECT. V.

If a Cone be any where cut by a Right-line, either parallel to its Axis, (as *SA*, or otherwiſe as *x N*.) ſo as the cutting Line being continu'd through one Side of the Cone, (as at *S*, or *x*) will meet with the other Side of the Cone, if it be continu'd or produc'd beyond the Vertex *V*, as at *T*. Then the Plane of that Section, (*viz.* the Figure *S b B B b S* is call'd an *Hyperbola*.



1. A Right-line being drawn through the Middle of any Hyperbola, *viz.* within the Section, (as *S a* or *x N*) is call'd the Axis or intercepted Diameter, (as in the Parabola) and that Part of it which is continu'd or produc'd out of the Section, until it meet with the other Side of the Cone continu'd, *viz.* *Ts*, or *T x* &c. is call'd the Tranſverſe-Diameter, or Tranſverſe-Axis of the Hyperbola.

2. All Right-lines that are drawn within an Hyperbola, at Right-Angles to its Axis, are call'd *Ordinates rightly apply'd*; as in the Ellipſis and Parabola.

3 That

3. That Ordinate which passes through the Focus of the Hyperbola, is call'd *Latus Rectum*, or *right Parameter*, for the same Reason as in the other Sections.

4. The middle Point of the Transverse-Diameter is call'd the Center of the Hyperbola, from whence may be drawn two Right-lines (out of the Section) call'd *Assymptotes*, because they will always incline (that is, come nearer and nearer) to both Sides of the Hyperbola, but never meet with (or touch) them, altho' both they and the Sides of the Hyperbola were infinitely extended, as will plainly appear in its proper Place.

These five Sections, *viz.* the Triangle, Circle, Ellipsis, Parabola, and Hyperbola, are all the Planes that can possibly be produc'd from a Cone. But of them the Three last are only call'd *Conic Sections*, both by the Antient and Modern Geometers.

SCHOLIUM.

Besides the foregoing Definitions, it may not be amiss to add, by Way of Observation, how one Section may (or rather doth) change or degenerate into another.

An Ellipsis being that Plane of any Section of the Cone, which is between the Circle and Parabola, it will be easy to conceive that there may be great Variety of Ellipses produc'd from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the Ellipsis change or degenerate into a Parabola. Now a Parabola being that Section whose Plane is always exactly parallel to the Side of the Cone, cannot vary as the Ellipsis may: For so soon as ever it begins to move out of that Position, (*viz.* from being parallel to the Cone's Side) it degenerates either into an Ellipsis, or into an Hyperbola. That is, if the Section inclines towards the Plane of the Cone's Base, it becomes an Ellipsis; but if it incline towards the Cone's Vertex, it then becomes an Hyperbola, which is the Plane of any Section that falls between the Parabola and the Triangle. And therefore there may be as many Varieties of Hyperbola's produc'd from one and the same Cone, as there may be Ellipses.

To be brief, a Circle may change into an Ellipsis, the Ellipsis into a Parabola, the Parabola into an Hyperbola, and the Hyperbola into a plane Isoceles Triangle. And the Center of the Circle, which is its Focus or Burning-Point, doth, as it were, part or divide it self into two Focus's so soon as ever the Circle begins to degenerate into an Ellipsis; but when the Ellipsis changes into a Parabola, one End of it flies open, and one of its Focus's vanishes,

vanishes, and the remaining Focus goes along with the Parabola when it degenerates into an Hyperbola. And when the Hyperbola degenerates into a plane Isosceles Triangle, this Focus becomes the Vertical Point of the Triangle, (*viz.* the Vertex of the Cone.) So that the Center of the Cone's Base may be truly said to pass gradually through all the Sections, until it arrives at the Vertex of the Cone, still carrying its *Latus Rectum* along with it. For the Diameter of a Circle being that Right-line which passes through its Center or Focus, and by which all other Right-lines drawn within the Circle are regulated and valued, may, (I presume) be properly call'd the Circle's *Latus Rectum*; and altho' it loses the Name of Diameter when the Circle degenerates into an Ellipsis, yet it retains the Name of *Latus Rectum*, with its first Properties in all the Sections, gradually shortening as the Focus carries it along from one Section to another, until at last it and the Focus becomes coincident, and terminate in the Vertex of the Cone.

I have been more particular, and full in these Definitions, than is usual in Books of this Subject, which I hope is no Fault, but will prove of Use, especially to a Learner; and altho' they may perhaps seem a little strange, and at first hard to be understood, yet when they are well consider'd and compar'd with a Cone, cut into such Sections as have been defin'd, they will not only be found true, but will also help to form a true and clear Idea of each Section.

CHAP. II.

Concerning the chief Properties of an Ellipsis.

Note. If the Transverse-Diameter of an Ellipsis, as TS in the following Figure, be intersected or divided into any two Parts by an Ordinate rightly apply'd, as at the Points A, C, a, &c. Then are those Parts, TA, TC, Ta, and SA, SC, Sa, &c. usually call'd Abscissa's, (which signifies Lines or Parts cut off) and by the Rectangle of any two Abscissa's, is meant the Rectangle of such two Parts as, being added together, will be equal to the Transverse-Diameter.

As $TA + SA = TS$. And $TC + SC = TS$.
Or $TA + SA = TS$, &c.

SECT.

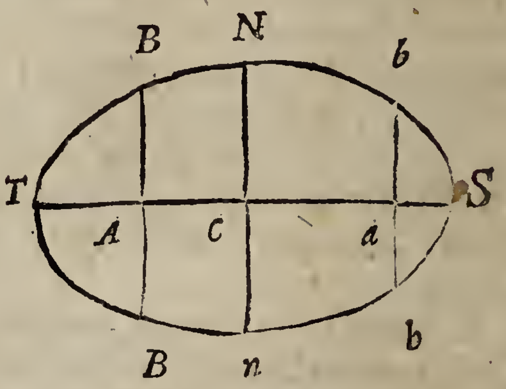
SECT. I.

Every Ellipsis is proportion'd, and all such Lines as relate to it, are regulated by the Help of one general *Theorem*.

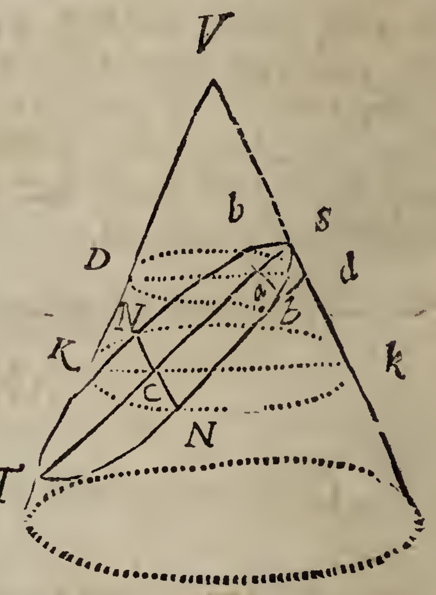
Theorem. *As the Rectangle of any two Abscissa's : is to the Square of Half the Ordinate which divides them : : So is the Rectangle of any other two Abscissa's : To the Square of half that Ordinate which divides them.*

That is,

$TA \times SA : \square BA : : Ta \times sa : \square ba$
$TA \times SA : \square BA : : TC \times SC : \square NC$
$TC \times SC : \square NC : : Ta \times Sa : \square ba$
&c.



Demonstration.



Let the annex'd Figure represent a right Cone, cut through both its Sides by the Right-line TS; then will the Plane of that Section be an Ellipsis, (by Sect. 3. *Chap.* 1.) $T\ S$ will be the Transverse-Diameter NCN and $b\,a\,b$ will be the Ordinates rightly apply'd. As before.

Again, if the Line Dd and Kk be parallel to the Cone's Base, they will be Diameters of Circles (by Sect. 2. Chap. 1.) Then will $\triangle\,TCK$ and TaD be alike. Also $\triangle\ Sad$ and $\triangle\ SCk$ will be alike.

Ergo	1	$Sa : ad : : SC : Ck.$	*Per* Theorem 13.
And	2	$TC : CK : : Ta : aD.$	
1 ∵	3	$Sa \times Ck = ad \times SC.$	
2 ∵	4	$Ta \times CK = TC \times aD.$	
2 × 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD.$	*Per* Axiom 3.
But	6	$CK \times Ck = \square\ NC$	*Per* Lemma Sect. 2.
And	7	$aD \times ad = \square\ ba$	
Then		for $CK \times Ck$, and $aD \times ad$, take $\square\ NC$, and $\square\ ba$	
5, 6, 7.	8	$Sa \times Ta \times \square\ NC = TC \times SC \times \square\ ba.$	*Per* Axiom 5.
Hence	9	$Sa \times Ta : \square\ ba : : TC \times SC : \square\ NC.$	See *Page* 194.

Q. E. D.

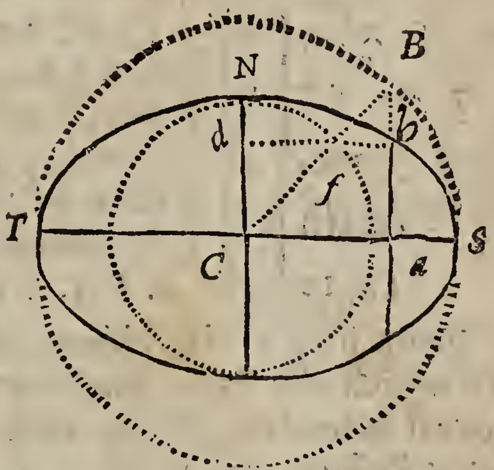
Or

Or the Truth of theſe Proportions may be otherwiſe prov'd by a Circle, without the Help of the Cone. Thus;

Let any Ellipſis be circumſcrib'd and inſcrib'd with Circles, as in the following Figure. Then from any Point in the circumſcribed Circle's Periphery, as at *B*, draw the Right-line *B a* parallel to the ſemi-conjugate Diameter *NC*, then will *b a* be a ſemi-ordinate rightly apply'd to the Tranſverſe Diameter *TS*, as before.

Again, from the Point *b* (in the Ellipſis's Periphery) draw the Right line *b d*, parallel to the Tranſverſe *TS*; and draw the Radius *BC*.

Then will △ *BCa* and △ *Cfd* be alike.



Therefore	1	{ *BC* : *Ba* : : *Cf* : *dC* Per *Theorem* 13.
But	2	{ *TC* = *BC* *NC* = *Cf* and *ba* = *dC*
Conſequ.	3	*TC* : *Ba* : : *NC* : *ba*
Or	4	*TC* : *NC* : : *Ba* : *ba*
4 in □'s	5	□*TC* □*NC* : : □*Ba* : □*ba*
But	6	{ *Ta* × *Sa* = □ *Ba* Per *Lem. Sect.* 2. *Ch.* 1.
Therefore	7	{ *Ta* × *Sa* : □ *ba* : : *TC* × *SC* = □*TC* : □ *NC*. As before.

And ſo for any other Abſciſſa's, and their Semi-ordinates.

Theſe Proportions being found to be the true and common Properties of every Ellipſis, all that is farther requir'd in (or about) that Section, may be eaſily deduc'd from them.

Sect. 2. *To find the* Latus Rectum, *Or* Right Parameter *of any Ellipſis.*

There are ſeveral Ways of finding the *Latus Rectum*; but I think none ſo eaſy, and ſhews it ſo plainly to be the third principal Line in the Ellipſis, as the following Theorem.

Theorem. { *As the Tranſverſe Diameter : is in Proportion to the Conjugate : : So is the Conjugate: To the* Latus Rectum.

Viz. (*in the following Figure*) *TS* : *Nn* : : *Nn* : *L R the* Latus Rectum.

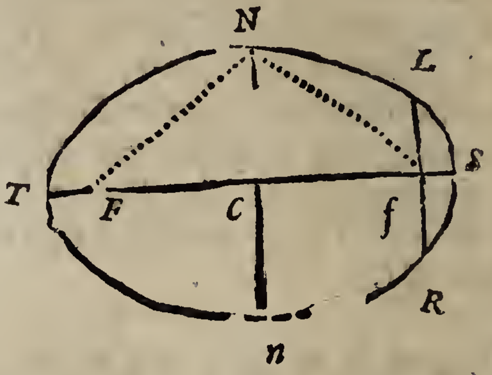
Demonſtration.

From the laſt Proportions take either of the Antecedents, and its Conſequent, *viz.* either *TC* × *SC* : □ *NC*. Or *Ta* × *Sa* : □ *ba*

 and

and make TS the third Term, to which find a fourth Proportional, and it will be $= LR$:

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\begin{cases} TC = SC \\ \text{And } NC = Cn \end{cases}$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5 ∴	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} Nn \times TS$
6 × 4	7	$\square TS \times LR = \square Nn \times TS$
7 ÷ ST	8	$\begin{cases} TS \times LR = \square Nn \\ \text{Which gives the following Analogy.} \end{cases}$
viz.	9	$TS : Nn :: Nn : LR$
Again	10	$\begin{cases} TC \times SC : \square NC :: Ta \times Sa : \square ba. \\ \text{By common Properties.} \end{cases}$
1, 10	11	$TS : LR :: Ta \times Sa : \square ba$



From hence 'tis evident, that LR thus found, is that Ordinate by which the other Ordinates may be regulated and found. Therefore (according to its Definition, *Sect.* 3. *Chap.* 1.) it is the true *Latus Rectum.* Q. E. D.

Consectary.

Hence it follows, that if the Transverse, and Conjugate Diameters of any Ellipsis are given (either in Lines or Numbers) the *Latus Rectum* may be easily found; and then any Ordinate, whose Distance from the Conjugate is given, may be found, as above.

Sect. 3. *To find the* Focus *of any* Ellipsis.

The Focus is the Distance of the *Latus Rectum* from the Conjugate or Middle of the Ellipsis, (*vide* Definition 4. Page 358.) And that Distance is always a mean Proportional between the Half Sum and Half Difference of the Transverse and Conjugate Diameters, which gives this Theorem.

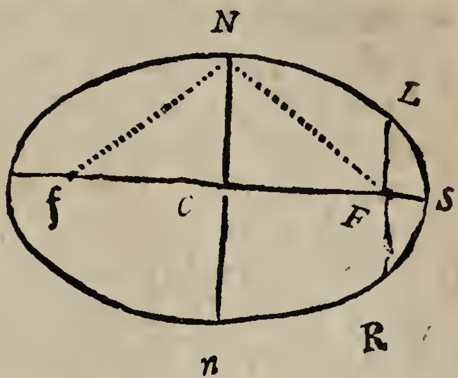
Theorem { *From the Square of Half the Transverse, subtract the Square of Half the Conjugate; the square Root of their Difference will be the Distance of each Focus from the middle or common Center of the Ellipsis.*

That is, supposing the Points f and F to be the two Focus's, viz. $fC = CF$, and $TC = \frac{1}{2} TS$. $NC = \frac{1}{2} Nn$. Then $TC + NC : fC :: FC : TC - NC$. Ergo $\square FC = \square TC - \square NC$. Consequently, $FC = \sqrt{\square TC - \square NC}$.

Demon

Demonstration.

First	1	$TS \times LR = \square Nn$. By 8 Step of the last Process.
And	2	$TS : LR :: TF \times SF : \square LF$. *Per* common Properties.
That is	3	$TS : LR :: TC + CF \times TC - CF : \frac{1}{4} \square LR = \square LF$
3 ∵	4	$\frac{1}{4} \square LR \times TS = \square TC - \square CF \times LR$
4 ÷ L R	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$
1 ÷ 4	6	$\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn = \square NC$
5, 6	7	$\square NC = \square TC - \square CF$
7, ±	8	$\square CF = \square TC - \square NC$
8 ω 2	9	$CF = \sqrt{\square TC - \square NC}$

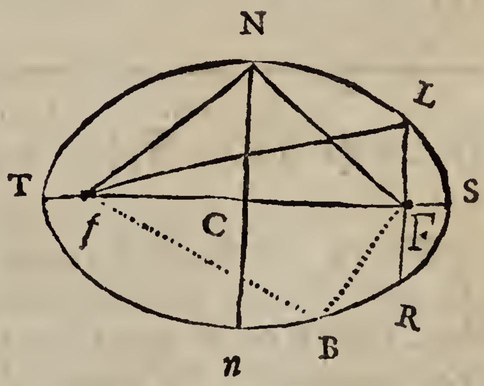


Now from hence is deduc'd that notable Proposition, upon which is grounded the usual Method of describing an Ellipsis, and drawing of Tangents, &c.

Proposition. *If from the two Focus's of any Ellipsis there be drawn two Right-lines, so as to meet each other in any Point of the Ellipsis's Periphery, the Sum of those Lines will be equal to the Transverse.*

Viz. $fN + NF = TS$. $fL + LF = TS$. Or $fB + BF = TS$, &c.

Demonstration.



First	1	$\square CF + \square NC = \square TC$ By 8th of the last.
But	2	$\square CF + \square NC = \square NF$ By Theorem 15.
1, 2	3	$\square NF = \square TC$ By Axiom 5.
3 ω 2	4	$NF = TC$
Hence		$2NF = 2TC = TS$
Again	5	$TS : LR :: TF \times FS : \square LF$. By common Properties.
Conseq.	6	$\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$
But		$\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$
Ergo	7	$TC : LF :: TC + CF \times TC - CF : \square LF$
7 ∵	8	$TC \times LF = \square TC - \square CF$
But	9	$\square fF + \square LF = \square fL$. By Theorem 11.
That is.	10	$4 \square CF + \square LF = \square fL$ for $2\,CF := fF$
8 × 4	11	$4 TC \times LF = 4 \square TC - 4 \square CF$
10 + 11	12	$4 \square TC + \square LF = 4 TC \times LF : + \square fL$
12 =	13	$4 \square TC : - 4 TC \times LF : + \square LF = \square fL$

13 *w* 2	14	$2TC - LF = fL$
14 + LF	15	$2TC = fL + LF$. But $2TC = TS$
Ergo	16	$fL \times Lf = TS$. Q. E. D.

And this Propoſition muſt needs hold true to every Point in the Ellipſis's Periphery, *viz.* at B, *&c.* as will evidently appear to any one who rightly conſiders, That

As a Thread juſt the Length of the Diameter of any Circle, having its two Ends ty'd together, and then mov'd about a Point in the Center, (*viz.* by making it a double Radius) will, by drawing another Point in its Extremity, deſcribe the Periphery of a Circle (*vide* Definition, *Page* 280.) Even ſo,

If a Thread, juſt the Length of the Tranſverſe Diameter (TS) having its two Ends ſo fix'd upon the two Focus's, (*f* and F) as that it may be mov'd about them, by drawing a Point in its Extremity, (*viz.* at its full Stretch) it will deſcribe the true Periphery of an Ellipſis.

Now although this eaſy Way of deſcribing, or, as uſually phras'd, drawing an Ellipſis, be mechanical, and known even to moſt Joyners, Carpenters, *&c.* yet it gives as complete and clear an Idea of that Figure, as any other Way whatſoever; and by deſcribing it thus about its two Focus's, as a Circle is about its Center, doth plainly ſhew thoſe two Points are not improperly call'd particular Centers in Definition 4. Sect. 3. Chap. 1. For each of them bears much the ſame Reſpect to the Ellipſis's Periphery, as the Circle's Center doth to its Periphery.

Sect. 4. *To deſcribe, or delineate an Ellipſis ſeveral Ways.*

There are ſeveral (other) Ways of deſcribing an Ellipſis, both Geometrically and Numerically, according to peculiar Occaſions; but I ſhall only mention two or three of them, leaving the reſt to the Learner's Genius. Now in order to that Work, it will be convenient to conſider what Lines are requiſite to limit or bound its Form, which I take to be chiefly theſe following.

1. If the Tranſverſe and Conjugate are given, the Ellipſis is perfectly limited, (*vide* Conſectary *Page* 363.) For if T S and N *n*, be ſet at Right-angles in their Middle at C, and T C or C S, be ſet off from N, or *n*, both Ways upon the Tranſverſe to *f* and F; (*viz.* make $fN = TC = NF$) then will thoſe Points *f* and F be the two Focus's, (by 4th Step of the laſt Proceſs) and then the Ellipſis may be deſcrib'd as above.

2. If

2. If the Transverse Diameter and *Latus Rectum* are given, the Ellipsis is truly limited, because by them the Conjugate may be found, by *Sect.* 2.

3. Or if only the Transverse, and the Proportion it hath either to the Conjugate or *Latus Rectum*, be given, the Ellipsis is thereby limited: As for Instance, suppose the given Ratio between the Transverse and Conjugate to be, As a : to d;

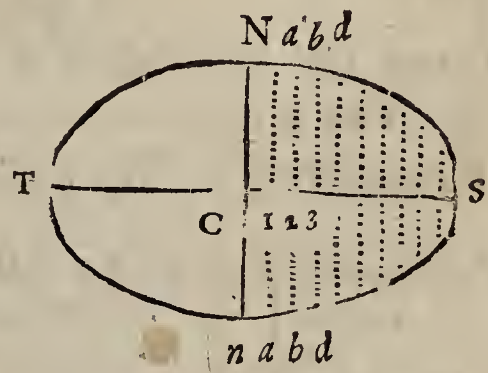
Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the Transverse, or Conjugate, and the Distance of the Focus from the Conjugate be given, the Ellipsis is limited, because by them the Conjugate or Transverse may be found.

These being premis'd, and the preceding Work a little consider'd, it must needs be easy to describe, or delineate any Ellipsis in *Plano*, either Geometrically or Numerically.

1. *To describe an Ellipsis Numerically by Points.*

Suppose the Transverse Diameter $TS = 20$, and the Conjugate $Nn = 12$, (either Inches or any other equal Parts) and let them cross each other at Right-angles in their Middles, As in the Point C.
Then will $TC = CS = 10$.
And $NC = Cn = 6$
And it will be $20 : 12 :: 12 : 7,2 =$ the *Latus Rectum*.



Again $20 : 7,2$ Or rather take their Ratio.

Thus $\begin{cases} 1 : 0,36 :: 10 + 1 \times 10 - 1 : \square a. \parallel 1. \\ 1 : 0,36 :: 10 + 2 \times 10 - 2 : \square b. \parallel 2. \\ 1 : 0,36 :: 10 + 3 \times 10 - 3 : \square d. \parallel 3. \end{cases}$ &c.

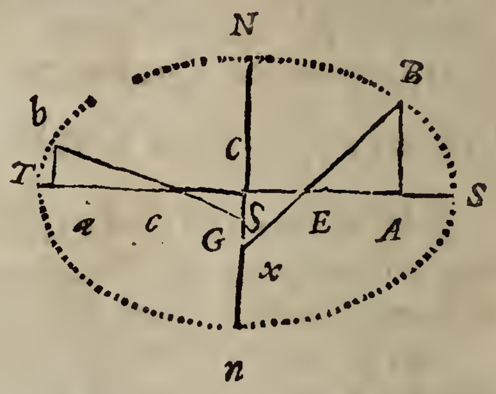
Viz. $\begin{cases} 100 - 1 \times 0,36 = \square a.1. \text{ Hence } \sqrt{99 \times 0,36} = 5,97 \text{ \&c.} = a.\ 1. \\ 100 - 4 \times 0,36 = \square b.2. \quad \sqrt{96 \times 0,36} = 5,88 \text{ \&c.} = b.\ 2. \\ 100 - 9 \times 0,36 = \square d.3. \quad \sqrt{91 \times 0,36} = 5,72 \text{ \&c.} = d.\ 3. \end{cases}$

If so many Semi-ordinates as may be thought convenient (the more the better) be found in this manner, and every one of them be set off at Right-angles from its respective Point in the Transverse Diameter each Way; *viz.* from 1 to *a*, from 2 to *b* from 3 *d*, &c. Then if a curve Line be carefully drawn with an even Hand, through those extreme Points *a*, *b*, *d*, &c. it will be the Ellipsis Periphery requir'd.

2. To

2. *To describe an Ellipsis Geometrically by Points.*

Having the Transverse and Conjugate Diameters given, *viz.* TS and Nn, plac'd at Right-angles in their Middles, as before; then from either End of the Conjugate, *viz.* N (or n) set off Half the Transverse Diameter to x. That is, make $Nx = TC$ (continuing the Conjugate Nn, when it is shorter than TC.) Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line Cx at Pleasure. Suppose it at G, and from that Point at G set off the Distance Cx to the Transverse, as at E, *viz.* make $GE = Cx$, and join the Points GE with a Right-line, produc'd so far beyond E, as to make $EB = NC$. Consequently, $GB = TC$.



Then, I say, that where-ever the Point G was taken between C and x, the Point B will just touch (or fall in) the Ellipsis's Periphery.

Demonstration.

Draw the Right-line BA perpendicular to TS. viz. *let BA be a Semi-ordinate rightly apply'd to the Transverse Diameter TS. Then $\triangle GCE$ and $\triangle BAE$ will be alike.*

Consequen.	1	$CE : AE :: EG : EB$. By Theorem 13.
1, And	2	$CE + AE : AE :: EG + EB : EB$. See Page 192.
But	3	$CE + AE = CA$. $EG + EB = TC$. And $EB = NC$.
Therefore	4	$CA : AE :: TC : NC$
6, in $\square$'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, $\because$	5	$\dfrac{\square CA \times \square NC}{\square TC} = \square AE$
But	6	$\square NC - \square AB = \square AE$
That is	7	$\square EB - \square AB = \square AE$
6, 7	8	$\dfrac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 $\times \square TC$	9	$\square CA \times \square NC = \square NC \times \square TC : - \square AB \times \square TC$
9 $\pm$	10	$\square NC \times \square TC : - \square CA \times \square NC = \square AB \times \square TC$
10. Analogy	11	$\square TC \quad \square CN :: \square TC - \square CA : \square AB$
That is	12	$TC \times CS : \square NC :: TC + CA \times TC - CA : \square AB$

Which is according to the common Properties of the Ellipsis. Therefore the Point B is truly found. Q. E. D.

Hence

Hence it follows, that if a convenient Number of ſuch Lines as *G E B*, be ſo drawn, as above directed, from the like Number of Points taken between *C* and *x*, &c. their extreme Points (as at *B*) will be thoſe Points by which (with an even Hand) the Ellipſis may be truly deſcribed, as before:

But if this be well underſtood, it will be very eaſy to conceive how to deſcribe an Ellipſis very readily, without drawing thoſe Lines, by having a thin, ſtrait, narrow Ruler, juſt the Length of *T C*, made ſomewhat ſharp at both Ends, upon which, from one of its Ends, ſet off the Length of *N C*. Then if the Point upon the Ruler which repreſents *E* be gradually or eaſily mov'd along the Tranſverſe *T S*, and at the ſame time the Point or End repreſenting *G* be kept ſliding cloſe along the Conjugate *Nn*, 'tis evident from the Work above, that the End of the Ruler repreſenting *B*, will, by that Motion, aſſign the true Periphery of the Ellipſis requir'd. For by that Motion the ſtrait Edge of the Ruler doth ſupply an infinite Number of the aforeſaid Lines; as will appear very plain and eaſy in Practice.

SCHOLIUM.

Now from hence was deduc'd the firſt Invention of that well-contriv'd Inſtrument for drawing an Ellipſis by one Motion, commonly call'd the Elliptical Compaſſes, being uſually made of Braſs, and compos'd of three Parts, two of which repreſent (or rather ſupply) the Tranſverſe and Conjugate Diameters ſet together at Right angles; and the third Part is a moveable Ruler, which performs the Office of the laſt mentioned thin Ruler. But becauſe the making of it is ſo well known to moſt Mathematical Inſtrument-Makers, eſpecially to that accurate and ingenious Artiſt Mr. *John Rowley*, Mathematical-Inſtrument-Maker, under St. *Dunſtan*'s-Church in *Fleet-ſtreet*, *London*; who for his great Skill in contriving, framing, and graduating all Kind of Mathematical-Inſtruments, may, I believe, be juſtly call'd one of the beſt Workmen of his Trade in *Europe*. I think it needleſs therefore to give a particular Deſcription of that Inſtrument.

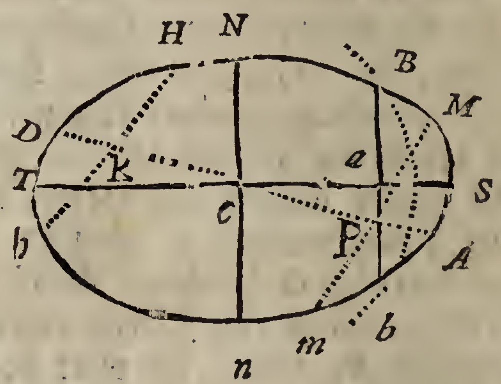
Alſo from hence came that ingenious Invention of making Engines for turning all Sorts of Elliptical or Oval Work, as Oval-Boxes and Picture-Frames, &c.

Sect.

Sect. 5. *Any Ellipsis being given, To find its* Transverse *and* Conjugate *Diameters.*

Suppose the given Ellipsis to be T N S *n* (in the annex'd Scheme) in which let it be requir'd to find the Transverse Diameter T S, and its Conjugate N *n*.

Draw within the Ellipsis any two Right-Lines parallel to each other, as H *b* and M *m*; and bisect those Lines, *viz.* find the middle Point of each, as at K and P. Then through those Points K and P, draw a Right-line, as D A, and it will be a Diameter; for it will divide the Ellipsis into two equal Parts. (See *Definit.* 1. *Page* 357.) Consequently the Middle of D A will be the true Middle or common Center of the Ellipsis, as at C.



For 'tis the Nature, or Property of all Diameters, howsoever they are drawn in any Ellipsis, (as it is in a Circle) to cut or cross one another in the common Center, or Middle of the Figure, as at C.

Upon the Point C describe an Arch of any Circle that will cut the Ellipsis Periphery in two Points, as at B and *b*; then joyn those Points B *b* with a Right-line, and it will be an Ordinate, through whose Middle, (as at *a*) and the common Center C, the Transverse Diameter T S must pass.

For B S = S *b*, and B *a* is at Right-angles with T S; therefore the Line B *b* is an Ordinate rightly apply'd to T S the Transverse Diameter. And if through the Point C there be drawn the Right-line N *n* parallel to B *b*, it will become the Conjugate; as was requir'd.

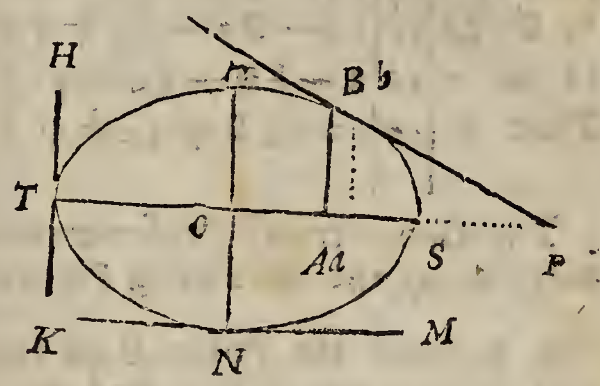
Sect. 6. *To draw a* Tangent, *or Right-line that may touch the Ellipsis's Periphery in any assign'd Point.*

The drawing of Tangents to or from any assign'd Point in the Ellipsis's Periphery, admits of three Cases.

CASE I. If it be requir'd to draw a Tangent that may touch the Ellipsis in either of the extreme Points of its Transverse Diameter, as at T or S, it is plain the Tangent must be drawn parallel to the Conjugate Diameter N *n*, as H K in the following Figure is suppos'd to be.

CASE

CASE 2. Or if the Tangent must be drawn to touch the Ellipsis in either of the extreme Points of its Conjugate Diameter, as at *N* or *n*, 'tis as evident that it must be drawn parallel to the Transverse Diameter T S, as K M. Consequently, if that Tangent, and the Transverse were both infinitely continu'd, they would never meet.



CASE 3. But if it be requir'd to draw a Tangent that may touch the Ellipsis in any other Point, as at *B*, &c.

Then if the Tangent and the Transverse-Diameter be both continu'd, they will meet in some Point, as at *P*; and those two Points (*viz.* *B* and *P*) do so mutually depend upon each other, that one of them must be assign'd in order to find the other, that so the Tangent may by them be truly drawn.

Let $D = TS$. $y = AS$: and $z = AP$. Then if y be given, z may be found by this *Theorem* $\left\{\frac{Dy - yy}{\frac{1}{2}D - y} = z\right.$

Or if z be given, y may be found by this *Theorem*:

Theorem. $\left\{\frac{D + z}{2} \pm \sqrt{\frac{DD + zz}{4}} = y\right.$

Demonstration.

Draw the Semi-Ordinate *b a*, as in the Figure; then will $\triangle$ *B A P* and $\triangle$ *b a P* be alike. Put $x = Aa$ the Distance between the two Semi-ordinates (viz. between *B A* and *b a*) which we suppose infinitely small.

Then	1	$z : z - x :: BA : ba$. By Theorem 13.
But	2	$\overline{D-y} \times y : \overline{D-y+x} \times \overline{y-x} :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2yx - Dx - xx :: \square BA : \square ba$
1 in $\square$'s.	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppose	5	$x = 0$ That so x may be every where rejected.
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8 ∴	9	$2yzz - Dzz = 2yyz - 2Dyz$
9 ÷ 2z	10	$yz - \frac{1}{2}Dz = yy - Dy$
10 ±	11	$\frac{1}{2}Dz - yz = Dy - yy$

11 ÷	12	$z=\dfrac{Dy-yy}{\frac{1}{2}D-y}$	Which is the 1st Theorem, and gives the following *Analogy.*
Analogy.	13	$\frac{1}{2}D-y : y :: D-y : z.$	Viz, $CA : SA :: TA : AP$
10 — yz	14	$yy-Dy-yz=-\frac{1}{2}Dz.$	
14 C □	15	$yy-Dy-yz+\frac{1}{4}DD-\frac{1}{2}Dz+\frac{1}{4}zz=\frac{1}{4}DD+\frac{1}{4}zz$	
15 ω 2	16	$y-\frac{1}{2}D-\frac{1}{2}z=\sqrt{\frac{1}{4}DD+\frac{1}{4}zz}$	
That is.	17	$y=\frac{1}{2}D+\frac{1}{2}z\pm\sqrt{\frac{1}{4}DD+\frac{1}{4}zz}$	Which is the 2d. Theor. Q.E.D.

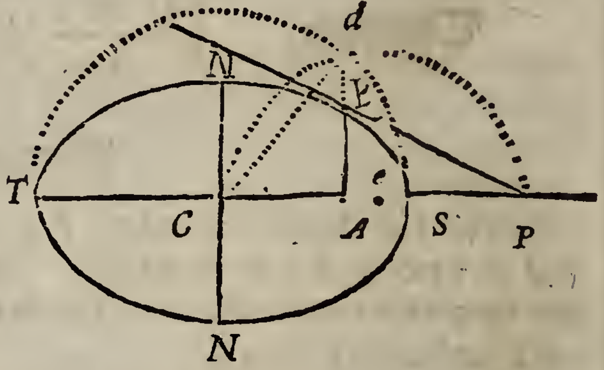
The Geometrical Performance of these two Theorems is very easy, as by the following Figure.

1. Suppose the Point B in the Ellipsis's Periphery were given, and it were requir'd to find the Point P, &c.

Make TC Radius, and upon the common Center C describe the Semi-circle TdS, and joyn the Points C and d with a Right-line; then bisect that Line, (by *Prob.* 2. *Page* 287.) and mark the Point where the bisecting Line would cross the Transverse, as at || e. Upon that Point || e, with the Radius Ce, (or ed) describe another Semi-circle. producing the Transverse-Diameter to its Periphery, and it will assign the Point P.

For if $D=TS.$ $y=AS.$ $z=AP.$ as before.

Then	1	$D-y\times y=\square dA$
And	2	$\frac{1}{2}D-y\times z=\square dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But		$CA=\frac{1}{2}D-y,$ &c.
1, 2	5	$\frac{1}{2}Dz-yz=Dy-yy$ As at 11 Step before.

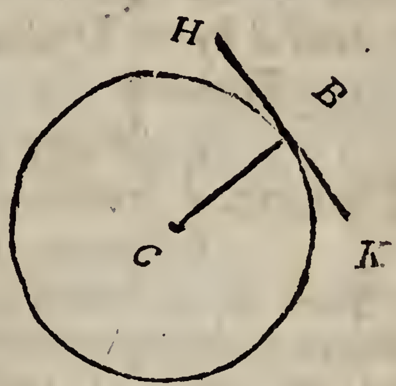


Therefore the Point P is truly found. Consequently, if a Right-line be drawn through those Points B and P, it will be the Tangent requir'd, according to the first Theorem.

2. The Converse of this is as easy; to wit, if the Point P be given, thence to find the Point B in the Ellipsis's Periphery. Thus, circumscribe Half the Ellipsis with the Semi-circle TdS, as before; and bisect the Distance between the Points C and P, as at || e, viz. let $Ce=eP$. Then making Ce Radius, upon the Point || e || describe the Semi-circle CdP; and from the Point where the two Semi-circles intersect or cross each other, as at || d, draw the Right-line dA perpendicular to the Transverse TS, and

and it will assign the Point of Contact *B* in the Ellipsis's Periphery through which the Tangent must pass.

But a Practical Method of drawing Tangents to any assign'd Point in the Ellipsis's Periphery, may (without finding the aforesaid Point *P*) be easily deduc'd from the following Property of Tangents drawn to a Circle, which is this.



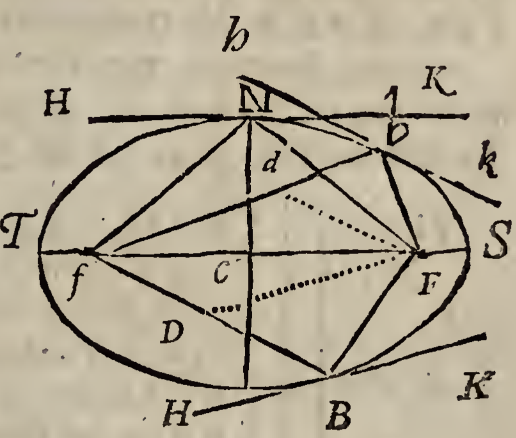
If to any Radius of a Circle, as *CB* there be drawn a Tangent-Line (as *HK*) to touch the Radius at the Point *B*, the two Angles which the Tangent makes with the Radius, will always be two Right-angles, (16, 17, 18, 19 *Euclid* 3.)

That is, $\angle HBC = \angle CBK = 90°$

In like manner the two Angles made between the Tangent and the two Lines drawn from the Focus's of an Ellipsis to the Point of Contact, will always be equal, but not Right-angles, save only at the two Ends of the Transverse-Diameter.

These being well consider'd, and compar'd with what hath been said in *Page 366*, it must needs be easy to understand the following Way of drawing Tangents to any assign'd Point in the Ellipsis's Periphery; which is thus:

Having by the Transverse and Conjugate-Diameters found the two Focus's *f*, and *F*, by *Sect.* 3. from them draw two Right-lines to meet each other in the assign'd Point of Contact, as *f b* and *Fb* (or *fB* and *FB*) in the annex'd Figure. Next set off (viz. make) $b\,d = b\,F$, (or $BD = BF$) and join the Points *F d* (or *F D*) with a Right-line.



Then, I say, if a Right-line be drawn through the Point of Contact *b* (or *B*) parallel to *d F*, or (*DF*) it will be the Tangent requir'd. For it is plain, that as the $\angle fNH = \angle FNK$ when the Tangent is parallel to the Transverse-Diameter, even so is the $\angle f\,b\,h = \angle Fb\,k$, (and $\angle fBH = \angle FBK$) and will be every where so, as the Point of Contact *b* (or *B*) and its Tangent, is carry'd about the Ellipsis's Periphery with the Lines *f b F* (or *f B F*)

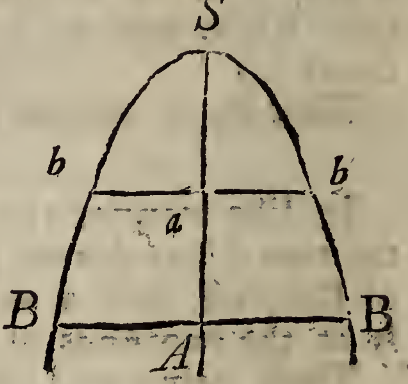
CHAP. III.

Concerning the chief Properties of every Parabola.

Note, In every Parabola, the intercepted Diameter, or that Part of its Axis which is between the Vertex and that Ordinate which limits its Length, as Sa or SA, &c. is call'd Abſciſſa.

Sect. 1. *The Plane or Figure of every Parabola, is proportion'd by its Ordinates and Abſciſſa's, as in the following Theorem.*

Theorem. *As any one Abſciſſa: is to the Square of its Semi-ordinate :: ſo is any other Abſciſſa: to the Square of its Semi-ordinate.*



That is, if we ſuppoſe the annex'd Figure to be a Parabola, wherein Sa, and SA are Abſciſſa's, and bab, BAB, Ordinates rightly apply'd, it will be $Sa : \square\, ba :: SA : \square\, BA$ / Or $Sa : SA :: \square\, ba : \square\, BA$ } whereſoever the Points a, A, are taken, And ſo for any other Abſciſſa, &c.

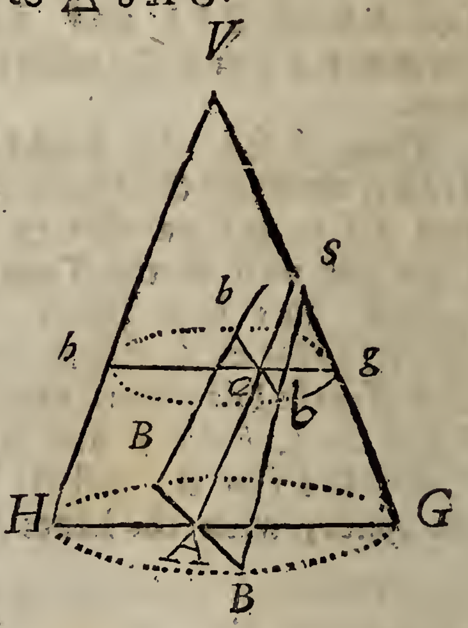
Demonſtration.

Let the following Figure HVG repreſent a Right-Cone, cut into two Parts by the Right-line SA, parallel to its Side VH. Then the Plane of that Section, viz. $BbSbB$ will be a Parabola by Sect. 4. Page 358. wherein let us ſuppoſe SA to be its Axis, and bab, BAB to be Ordinates rightly apply'd to that Axis.

Again, imagine the Cone to be cut by the Right-line bg parallel to its Baſe HG. Then will bg be the Diameter of a Circle, by Sect. 2. Page 357. And $\triangle\, Sag$ like to $\triangle\, SAG$.

Therefore	1	$Sa : ag :: SA : AG$ By Theorem 13.
1 ∵	2	$Sa \times AG = SA \times ag$
2 × ba	3	$Sa \times AG \times ba = SA \times ag \times ba$ By Axiom 3.
But	4	$HA = ba$ becauſe SA is parallel to VH
And	5	$\square\, BA = AG \times HA$ / $\square\, ba = ag \times ba$ } By Lem. P. 357.
3, 4, 5,	6	$SA \times \square\, BA = SA \times \square\, ba$ By Axiom 5.
6, Analogy	7	$Sa : \square\, ba :: SA : \square\, BA$ Vide Page 194.

—Q. E. D.



These

These Proportions being prov'd to be the common Property of every Parabola, all that is farther requir'd about that Section, or Figure, may from thence be easily deduc'd.

Sect. 2. *To find the* Latus Rectum, *or Right* Parameter *of any Parabola.*

The *Latus Rectum* of a Parabola hath the same Ratio or Proportion to any Abscissa, and its Semi-ordinate, as the *Latus Rectum* of an Ellipsis hath to its Transverse and Conjugate Diameters, and may be found by this Theorem.

Theorem. *As any Abscissa : is in Proportion to its Semi-ordinate : : so is that Semi-ordinate to the* Latus Rectum.

Let $L =$ the *Latus Rectum.*

Then	1	$Sa : ba :: ba : L$	wherever the Points a, and A, are taken in the Axis.
And	2	$Sa : BA :: BA : L$	
1 $\because$	3	$\frac{\square ba}{Sa} = L$:	Or $Sa \times L = \square ba$
2 $\because$	4	$\frac{\square Ba}{Sa} = L$:	Or $SA \times L = \square BA$
$3 = 4$	5	$\frac{\square BA}{SA} = \frac{\square Sa}{ba}$	Per Axiom 5.
$5 \times$	6	$Sa \times \square BA = Sa \times \square ba$	Which gives this
Analogy	7	$Sa : \square ba :: SA : \square BA$	The same as the 7th Step of

the last Process; therefore L (thus found) is the true *Latus Rectum*, by which all the Ordinates may be regulated and found, according to its Definition in Section 4. *Page* 358.

For by the 3d Step $Sa \times L = \square ba$. And by the 4th Step $SA \times L = \square BA$. Consequently, $\sqrt{Sa \times L} = ba$ And $\sqrt{SA \times L} = \square BA$; and so for any other Ordinate.

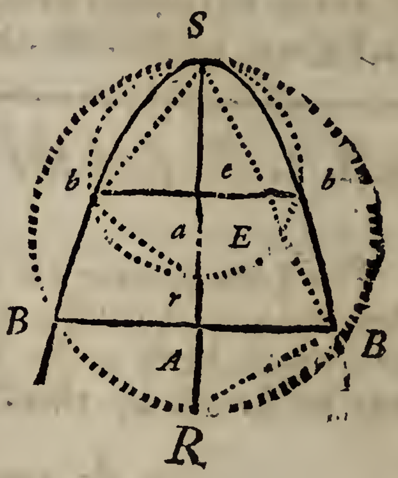
Or if the Ordinates are given, to find their Abscissa's; then it will be, $L : ba :: ba : Sa$. And $L : BA :: BA : SA$, &c.

Consequently, $\frac{\square ba}{L} = Sa$. And $\frac{\square BA}{L} = SA$, &c.

From the Consideration of these Proportions, it will be easy to conceive how to find the *Latus Rectum* Geometrically, thus:

Join

Join the Vertical Point S of the Axis, and either extreme Point of any Ordinate, as B, (or b) with a Right-line, viz. SB (or Sb) and bisect that Line, (by *Problem* 2. *Page* 287.) marking the Point where the bisecting Line doth intersect or cross the Axis, as at E, (or e) and with the Radius SE (or Se) upon the Point E (or e) describe a Circle; (as in the annex'd Figure) then will the Distance between the Ordinate and that Point where the Circle's Periphery cuts the Axis, viz. AR (or ar) be the true *Latus Rectum* requir'd.



For $Sa : BA :: BA : AR$. And $Sa : ba :: ba : ar$. By *Theor.* 13 Therefore $AR = L$. And $ar = L$ by the 1st and 2d Steps above.

CONSECTARY.

From these Proportions of finding the *Latus Rectum*, it will be easy to deduce and demonstrate this following Theorem.

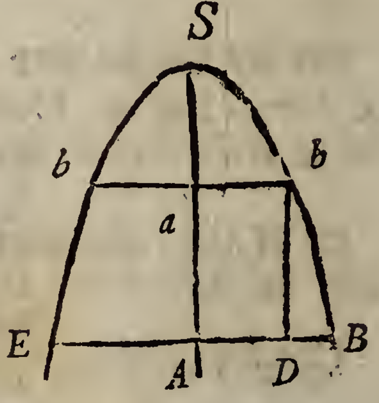
Theorem. *As the* Latus Rectum: *Is to the Sum of any two Semi-ordinates :: so is the Difference of those two Semi-ordinates: to the Difference of their Abscissa's.*

Suppose any Right-line drawn within the Parabola, as bD, parallel to its Axis SA; then will that Line (*viz.* bD) be a Diameter (by *Def.* 5. *Page* 359.) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be

$L : ED :: DB : bD$ according to the Theorem,

Demonstration.

First	1	$SA = \frac{\square BA}{L}$ By Step 2. of the last Process.
And	2	$Sa = \frac{\square ba}{L}$ By Step 1. of the last Process.
1 − 2	3	$SA - Sa = \frac{\square BA - \square ba}{L}$
3 × L	4	$\overline{SA - Sa} \times L = \square BA - \square ba$
But	5	$\square BA - \square ba = \overline{BA + ba} \times \overline{BA - ba}$
4 = 5, 6	6	$\overline{SA - Sa} \times L = \overline{BA + ba} \times \overline{BA - ba}$
6, Analogy	7	$L : BA \times ba :: BA - ba : SA - Sa$
Or	8	$L : ED :: DB : bD$



Which gives the following Analogy.

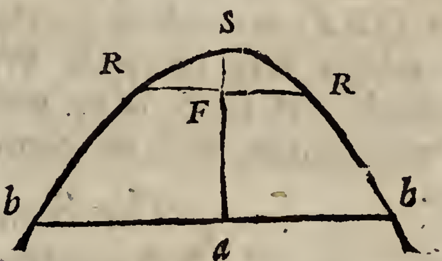
This

This peculiar Property of the Parabola was first publish'd *Anno* 1684. by one Mr. *Thomas Baker*, Rector of *Bishop Nympton* in *Devonshire*, in a Treatise intitled, *The Geometrical Key*: or, *The Gate of Æquations unlock'd*; wherein he hath shew'd the Geometrical Construction and Solution of all Cubic and Biquadratic affected Æquations by one general Method, which he calls a *Central Rule*, deduc'd from this peculiar Property of the Parabola.

Sect. 3. *To find the* Focus *of any* Parabola.

The Focus of every Parabola is that Point in its Axis through which the *Latus Rectum* doth pass. (See *Definition* 3. *Sect.* 4. *Page* 359.) Therefore its Distance from the Vertex of the Parabola may be easily found, either by the *Latus Rectum* it self, or by any other Ordinate, and its Abscissa.

Thus, suppose the Point at F to be the Focus, S the Vertex, the Ordinate $RFR = L$ the *Latus Rectum*, and bab any other Ordinate.



Then will $SF = \frac{1}{4}L$. Or $SF = \frac{\square ba}{4Sa}$

Demonstration.

First	1	$SF \times L = \square FR$ By *Sect.* 2. *Page* 375.
And	2	$FR = \frac{1}{2}L$. For the Ordinate $RFR = L$ as above.
2 Ⓢ 2	3	$\square FR = \frac{1}{4}\square L = \frac{1}{2}L \times \frac{1}{2}L$
1, = 3	4	$SF \times L = \frac{1}{4}\square L$
4 ÷ L	5	$SF = \frac{1}{4}L$ As by *Definition* 4. *Sect.* 4. *Page* 359.
Again	6	$\frac{\square ba}{Sa} = L$ by the third Step in *Page* 375.
Conseq.	7	$\frac{\square ba}{4Sa} = \frac{1}{4}L$. &c. As above.

Q. E. D.

Sect. 4. *To* Describe, *or Draw a Parabola several Ways.*

Note, *There are two or three Ways of drawing a Parabola Instrumentally at one Motion; but because those Instruments or Machines are not only too perplex'd for a Learner to manage, but also a little subject to Error, I have therefore chosen to shew how that Figure may be (the best) drawn by a convenient Number of Points,* viz. *Ordinates found, either Numerically or Geometrically, according to the* Data; *which, if the Work of the three last Sections be well consider'd, must needs be very easy.*

1. If

1. If any Ordinate and its Abſciſſa are given, there may by them be found as many Ordinates as you pleaſe to aſſign or take Points in the Parabola's Axis, (by *Sect.* 1. *Page* 374.) and the Curve of the Parabola may be drawn by the extreme Points of thoſe Ordinates, as the Ellipſis was *Page* 367.

2. If the *Latus Rectum*, and either any Ordinate, or its Abſciſſa, are given, then any aſſign'd Number of Ordinates may by them be found, (by *Sect.* 2. *Page* 375,) either Numerically or Geometrically, *&c.*

3. If only the Diſtance of the Focus from the Vertex of the Parabola be given, any aſſign'd Number of Ordinates may be found by it. For S F $= \frac{1}{4}$ L the *Latus Rectum*, and $\frac{1}{2}$ L = F R as in the laſt *Section*; and it will be, as S F : is to □ F R : : ſo is any other Abſciſſa (viz. *Sa*, or *SA*, &c.) : to the Square of its Semi-ordinate, (viz. □*ba*; or □*BA*) according to the common Property of the Parabola.

Altho' any of theſe Ways of finding the Ordinates are eaſy enough, yet that Way which may be deduc'd from the following Propoſition, will be found much more eaſy, and ready in Practice.

Propoſition. *The Sum of any Abſciſſa and focal Diſtance from the Vertex, will be equal to the Diſtance from the Focus to the extreme Point of the Ordinate which cuts off that Abſciſſa.*

For Inſtance, Suppoſe *S* to be the Vertex of any Parabola, the Point *F* to be its Focus, and *AB* any Semi-ordinate rightly apply'd to its Axis *SA*. Then I ſay, where-ever the Point *A* is taken in the Axis, it will be *SA* + *SF* = *FB*. Conſequently, if *Sf* = *SF*, it will be *fA* = *FB*.

Demonſtration.

Firſt	1	$SF = \frac{1}{4}L$ by the 7th Step, *Sect.* 3.
Ergo	2	$fA = FA + \frac{1}{2}L$ by Conſtruction above.
2 ⊛ 2	3	$\square fA = \square FA + \overline{FA \times L} : \frac{1}{4}LL$
Again	4	$SA = FA + \frac{1}{4}L$ by the Suppoſition and Figure.
4 × L	5	$SA \times L = \overline{FA \times L} : + \frac{1}{4}LL$ But $SA \times L = \square AB$
Ergo	6	$\square AB = \overline{FA \times L} : + \frac{1}{4}LL$
3 − 6	7	$\square fA - \square AB = \square FA$. Conſe. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$. By *Theorem* 11.
Ergo	9	$\square fA = FB$
9 ⱳ 2	10	$fA = FB$ Q. E. D.

This

This Proposition being well understood, it will be very easily apply'd to Practice, supposing the Focal Distance given, or any other Data by which it may be found.

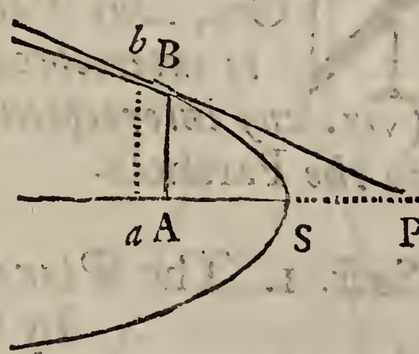
Thus draw any Right Line to represent the Parabola's Axis, and from its Vertical Point, as at S, set off the Focal Distance both upwards and downwards, *viz.* make $Sf = SF$ the Distance of the given Focus from the Vertex, as in the last Scheme Then by the Proposition it is evident, that if never so many Lines be drawn ordinately at Right Angles to the Axis the true Distance between the Point f out of the Parabola, and any of those Lines (or Ordinates) being measur'd or set off from the Focus F to the same Line (or Ordinate) it will assign the true Point in that Line through which the Curve must pass. That is, it will shew the true Limits or Length of that Ordinate, as at B in the last Scheme.

Proceeding on in the very same Manner, from Ordinate to Ordinate, you may with great Expedition and Exactness find as many Ordinates (or rather their Points only, like B) as may be thought convenient; which being all joyn'd together with an even Hand, will form the Parabola requir'd.

N.B. The more Ordinates (or their Points) there are found, and the nearer they are to one another, the easier and exacter may the Curve of the Parabola be drawn. The same is to be observ'd when any other Curve is requir'd to be drawn by Points.

Sect. 5. *To draw a* Tangent *to any given* Point *in the* Curve *of a* Parabola.

Tangents are very easily drawn to the *Curve* of any *Parabola*.



For, Supposing S to be its Vertex, B the Point of Contact, *viz.* the Point where the Tangent must touch the Curve. And P the Point where the Tangent will intersect, (or meet with) the Parabola's Axis produc'd. Then if from the Point of Contact B, there be drawn the Semi-ordinate BA at Right Angles to the Axis SA, wheresoever the Point A falls in the Axis, it will be $SP = SA$.

Demonstration.

Draw the Semi-ordinate ba (as in the Figure) then will the $\triangle BAP$, and $\triangle baP$ be alike. Let $y = AS$ the Abscissa, and

 $z = SP$;

$z = SP$; put $x = Aa$ the Distance between the two Semi-ordinates; which we suppose to be infinitely near each other, as in the Ellipsis, *Pag.* 371.

Then	1	$y+z : BA :: y+z+x : ba$. Per *Theorem* 13.
1, Or	2	$y+z : y+z+x :: BA : ba$. See *Page* 192.
Again	3	$y : \square BA :: y+x : \square ba$. Per *Theorem Page* 374.
3, Or	4	$y : y+x :: \square BA : \square ba$
2 in $\square$'s	5	$yy+2yz+zz : yy+2yz+2yx+zz+2zx+xx :: \square BA : \square ba$
4, 5	6	$y : y+x :: yy+2yz+zz : yy+2yz+2yx+zz+2zx+xx$
6 ∴	7	$yy+2yz+yx+zz+2zx+\frac{zzx}{y} = yy+2yz+2yx+zz+2zx+xx$
That is,	8	$\frac{zzx}{y} = yx+xx$. Consequently $\frac{zz}{y} = y+x$
Suppose	9	$x = 0$ And rejected, as in the Ellipsis, *Page* 371.
Then	10	$\frac{zz}{y} = y$ Consequently $zz = yy$
10 ω 2	11	$z = y$ That is, $SP = SA$ Q.E.D

CHAP. IV.

Concerning the chief Properties of the Hyperbola.

NOte, *Any Part of the Axis of an* Hyperbola, *which is intercepted between its Vertex and any Ordinate,* (viz. *any intercepted Diameter*) *is call'd an* Absciffa, *As in the* Parabola.

Sect. 1. The *Plane* of every *Hyperbola* is proportion'd by this *General Theorem.*

Theorem. *As the Sum of the Transverse and any* Abfciffa *multiply'd into that* Abfciffa: *is to the Square of its Semi-ordinate* :: *so is the Sum of the Transverse and any other* Abfciffa *multiply'd into that* Abfciffa : *To the Square of its Semi-ordinate.*

That

That is, if TS be the Tranſverſe Diameter,

And $\begin{cases} Sa,\ SA, & \text{Abſciſſa's.} \\ ba,\ BA, & \text{Semi-ordinates.} \end{cases}$

Then is $\begin{cases} Ta = TS + Sa. \\ TA = TS + SA. \end{cases}$

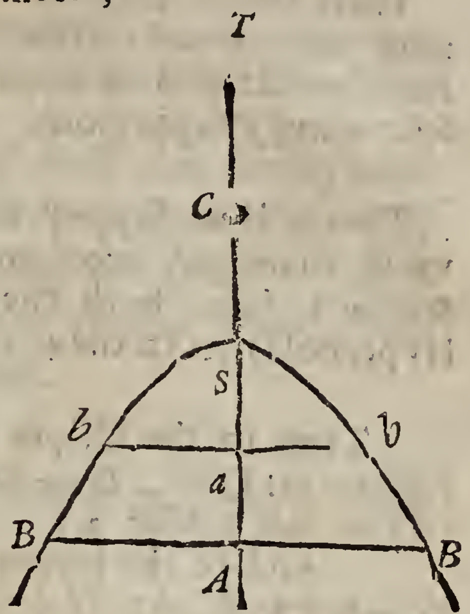
And it will be

$Ta \times Sa : \square\, ba :: TA \times SA : \square\, BA.$

That is,

$TS + Sa \times Sa : \square\, ba :: TS + SA \times SA : \square\, BA$

&c.

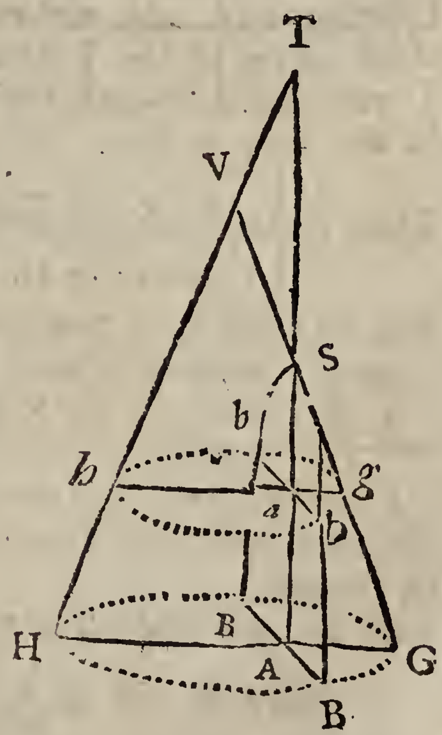


Demonſtration.

Let the following Figure H V G repreſent a Right Cone cut into two Parts by the Right Line S A; then will the Plane of that Section be an Hyperbola (by *Sect.* 5. *Chap.* 1.) in which let S A be its Axis, or intercepted Diameter, bab, and B A B, Ordinates Rightly apply'd (as before in the Parabola) and T S its Tranſverſe Diameter.

Again, if the Cone is ſuppos'd to be cut by hg, parallel to its Baſe H G, it will alſo be the Diameter of a Circle, &c. as in the Ellipſis and Parabola.

Then will the $\triangle Sga$ and $\triangle SGA$ be alike, alſo the $\triangle Tab$ and TAH will be alike; therefore it

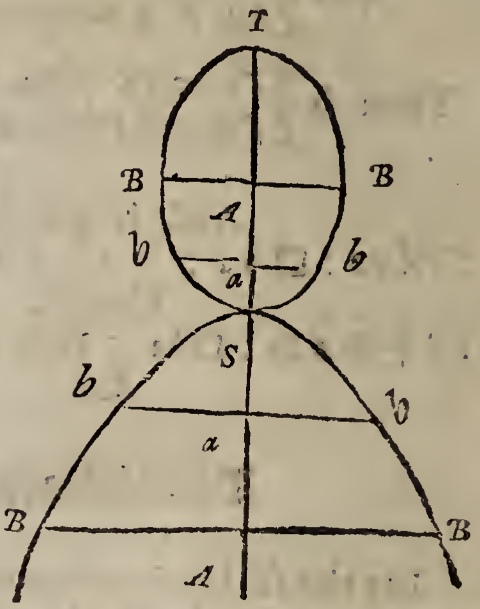


will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ah :: TA : AH$
1 ∴	3	$Sa \times AG = SA \times ag$
2 ∴	4	$Ta \times AH = TA \times ah$
3 × 4	5	$\begin{cases} Sa \times Ta \times AG \times AH = \\ = SA \times TA \times ag \times ah \end{cases}$
But	6	$ag \times ah = \square\, ab$
And	7	$\begin{cases} AG \times AH = \square\, AB \\ \text{Per } \textit{Lemma Page } 357. \end{cases}$
5, 6, 7	8	$\begin{cases} Sa \times Ta \times \square\, AB = \\ SA \times TA \times \square\, ab \\ \text{Which give the following} \end{cases}$
8, *Anal.*	9	$a \times Ta : \square\, ab :: SA \times TA : \square\, AB$, &c.

Q. E. D.

These

These Proportions are the common Property of every Hyperbola, and do only differ from those of the Ellipsis in the *Signs* + And — As plainly appears in all the following Proportions.



That is, if we suppose TS, the Transverse Diameter, common to both Sections, (*viz.* both the Ellipsis and Hyperbola) As in the annex'd Scheme.

Then in the Ellipsis it will be
$TS - Sa \times Sa : \Box ab :: TS - SA \times SA : \Box AB$
(As by *Sect.* 1. *Chap.* 2.)
And in the Hyperbola it is
$TS + Sa \times Sa : \Box ab :: TS + SA \times SA : \Box AB$
As above.

And therefore all that is farther requir'd in the Hyperbola, may (in a manner) be found as in the Ellipsis, due Regard being had to to changing of the Signs.

Sect. 2. *To find the* Latus Rectum, *or* Right Parameter *of any Hyperbola.*

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times Sa : \Box ab$. Or, $TA \times SA : \Box AB$. To them bring in the Transverse TS for a third Term, and by those Three find a Fourth Proportional (as in the Ellipsis) and that will be the *Latus Rectum.*

Thus	1	$Ta \times Sa : \Box ab :: TS : \dfrac{\Box ab \times TS}{Ta \times Sa} =$ the *Latus Rectum,* which call L (as in the Parabola.)
Then	2	$TS : L :: Ta \times Sa : \Box ab.$
But	3	$Ta \times Sa : \Box ab :: TA \times SA : \Box AB.$ Therefore
2, 3	4	$TS : L :: TA \times SA : \Box AB$, &c.

Consequently L is the true *Latus Rectum*, or Right Parameter, by which all the Ordinates may be found, according to its Definition in *Chapter* 1.

And because $TS + Sa = Ta$. Let it be $TS + Sa$, instead of Ta.

Then it will be $\dfrac{\Box ab \times TS}{TS \times Sa : + \Box Sa} = L$

And in the Ellipsis it would be $\dfrac{\Box ab \times TS}{TS \times Sa : - \Box Sa} = LR = L.$

Sect.

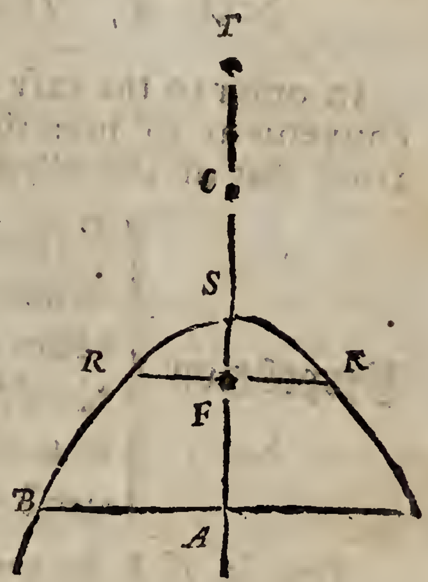
Sect. 3. *To find the* Focus *of any Hyperbola.*

The Focus being that Point in the Hyperbola's Axis, through which the *Latus Rectum* must pass, (As in the Ellipsis and Parabola) it may be found by this *Theorem*.

Theorem. *To the Rectangle made of half the Transverse into half the Latus Rectum, add the Square of half the Transverse; the Square Root of that Sum will be the Distance of the Focus from the Center of the Hyperbola.*

Demonstration.

Suppose the Point at *F*, in the annex'd Scheme, to be the Focus sought. Then will $FR = \frac{1}{2}L$.



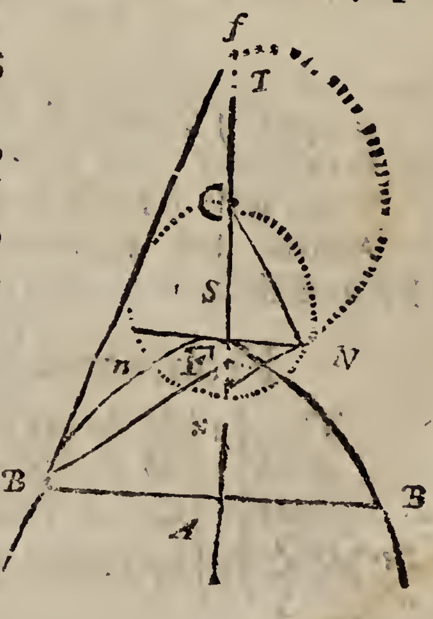
Let $TC = CS$ be half the Transverse; then is the Point *C* call'd the Center of the Hyperbola (for a Reason that shall be hereafter shew'd.) and $SF = a$.

Again, Let $CS = d$. And $SF = a$

Then	1	$2d : L :: 2d + a \times a : \frac{1}{4}LL$
That is,	2	$TS : L :: TS + SF \times FS : \square FR$
1 ∴	3	$\frac{1}{2}dL = 2da + aa$
3 + dd	4	$dd + \frac{1}{2}dL = dd + 2da + aa$
4 ɯ 2	5	$\sqrt{dd + \frac{1}{2}dL} = d + a = FC$:
Or, 5, — d	6	$\sqrt{dd + \frac{1}{2}dL} - d = a = SF$

In the Ellipsis it is, $2d : L :: 2d - a \times a : \frac{1}{4}LL$.
That is, $\frac{1}{2}dL = 2da - aa$. &c.

The Geometrical Effection of the last *Theorem* is very easily perform' d, thus;



Make $Sx = \frac{1}{2}L$, viz. half the *Latus Rectum*; and let $CS = d$, as above.

Upon *Cx* (as a Diameter) describe a *Circle*; and at *S* the Vertex of the Hyperbola draw the Right Line *nSN* at Right Angles to *Cx*; then join the Points *CN* with a Right Line, and 'twill be $CN = d + a = FC$.

For	1	$CS : SN :: SN : Sx$ per Fig.
That is,	2	$d : SN :: SN : \frac{1}{2}L$
2 ∴	3	$\frac{1}{2}dL = \square SN$
But	4	$dd + \square SN = \square CN$
3, 4	5	$dd + \frac{1}{2}dL = \square CN$
5 ɯ 2	6	$\sqrt{dd + \frac{1}{2}dL} = CN = d + a$, &c.

Now

Now here is not only found the Distance of the Hyperbola's Focus, either from its Center C, or Vertex S; but here is also found that Right Line usually call'd its Conjugate Diameter, viz. the Line n S N, which bears the same Proportion to the Transverse and *Latus Rectum* of the Hyperbola, as the Conjugate Diameter of the Ellipsis doth to its Transverse, and *Latus Rectum*.

For in the Ellipsis T S : N n :: N n : L R. *per Sect.* 2. *Page* 363.

Consequently $\frac{1}{2}$ T S : $\frac{1}{2}$ N n :: $\frac{1}{2}$ N n : $\frac{1}{2}$ L R.

But $\frac{1}{2}$ T S $= d$. $\frac{1}{2}$ N $n =$ S N. And $\frac{1}{2}$ L R $= \frac{1}{2}$ L.

Therefore d : S N :: S N : $\frac{1}{2}$ L. As at the second Step above.

What Use the aforesaid Line n S N is of, in Relation to the Hyperbola, will appear farther on.

Sect. 4. *To Describe an Hyperbola in Plano.*

In order to the easy describing of any *Hyperbola in Plano*, it will be convenient to premise the following Proposition, which differs from that of the Ellipsis in *Sect.* 3. *Chap.* 2. only in the Signs.

Proposition. *If from the Focus's of any Hyperbola, there be drawn two Right Lines, so as to meet each other in any Point of the Hyperbola's Curve, the Difference of those Lines* (in the Ellipsis it is their Sum) *will be equal to the Transverse Diameter.*

That is, if F be the Focus, and it be made C $f =$ C F, (as in the last Scheme) then the Point f is said to be a Focus out of the Section (or rather of the opposite Section) and it will be f B — F B $=$ T S.

Demonstration.

Suppose fC, or $CF = z$, and $SA = x$. Let CS, or $TC = d$, as before.

Then will $fA = d + x + z$, And $FA = d + x - z$.

Again, Let $FB = b$, and $fB = b$. Then $2d = b - b$, by the Proposition.

From these *substituted Letters* it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square FA$
But		$\square fA + \square AB = \square fB$. And $\square fA + \square AB = FB$
Per 4th of last	3	$dd + \frac{1}{2}dL = dd + 2da + aa = \square FC = zz$.

3 — dd

3 — dd	4	$zz - dd = \frac{1}{2}dL$
4 ÷ $\frac{1}{2}d$	5	$\frac{zz-dd}{\frac{1}{2}d} = L$
Again	6	$2d : L :: 2d + x \times x : \square AB$ By *com. Properties.*
5, 6	7	$2d : \frac{zz-dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$
7 ∴	8	$\frac{2dzzx + zzxx - 2dddx - ddxx}{dd} = \square AB$
1 + 8	9	$\left\{ dd + 2dx + 2dz + xx + 2zx + zz + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} \right. = \square fA + \square AB = bb$
2 + 8	10	$\left\{ dd + 2dx - 2dz + xx - 2zx + zz + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} \right. = \square FA + \square AB = bb$
9 × dd	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzxx = ddbb$
10 × dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzxx = ddbb$
11 ω 2	13	$dd + dz + zx = db$
12 ω 2	14	$dd - dz - zx = db$
13 ÷ d	15	$d + z + \frac{zx}{d} = b$
14 ÷ d	16	$d - z - \frac{zx}{d} = b$
16, or	17	$z + \frac{zx}{d} - d = b$
15 — 17	18	$2d = b - b$

Altho' the Æquation at the 16th Step be in it self impossible, because z is greater than d; (by the 4th Step) yet from thence it will be easy to conclude, that the Difference between d and $z + \frac{zx}{d}$ will give the true Value of b, as in the 17th Step.

But because I would leave no Room for the Learner to doubt about changing the Æquation, $d - z - \frac{zx}{d} = b$ into that of $z + \frac{zx}{d} - d = b$, it may be convenient to illustrate the whole Process in Numbers, whereby (I presume) it will plainly appear that $b - b = TS$.

In order to that. Let the Transverse $TS = 2d = 12$. Then $d = 6$ Suppose the Abscissa $SA = x = 4$. And the Semi-ordinate $AB = 3$

First	1	TS + SA × SA : □AB :: TS : L *per Sect.* 2.
1, Viz.	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1{,}6875 = L$
Again	3	$\sqrt{dd + \frac{1}{2}dL} = d + a = CF$ *per Sect.* 3.
3, Viz.	4	$\sqrt{36 + 5{,}0625} = 6{,}408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6{,}408 = 16{,}408 = fA$
And	6	$d + x - z = 6 + 4 - 6{,}408 = 3{,}592 = FA$

5 ☉ 2	7	$269{,}2224 = \square fA$
6 ☉ 2	8	$12{,}9024 = \square FA$
But	9	$9 = \square AB$ For $AB = 3$ by Suppoſition.
7 + 9	10	$278{,}2224 = \square fA + \square AB = \square fB$
8 + 9	11	$21{,}9024 = \square FA + \square AB = \square FB$
10 ω 2	12	$16{,}68 = fB$
11 ω 2	13	$4{,}68 = FB$
12 — 13	14	$12.00 = fB - FB = TS$. Which was to be prov'd

If this Propoſition be truly underſtood, it muſt needs be eaſy to conceive how to deſcribe the Curve of any Hyperbola very readily by Points, when the Tranſverſe Diameter and the Focus are given, (or any other Data by which they may be found, As in the preceding Rules.) Thus,

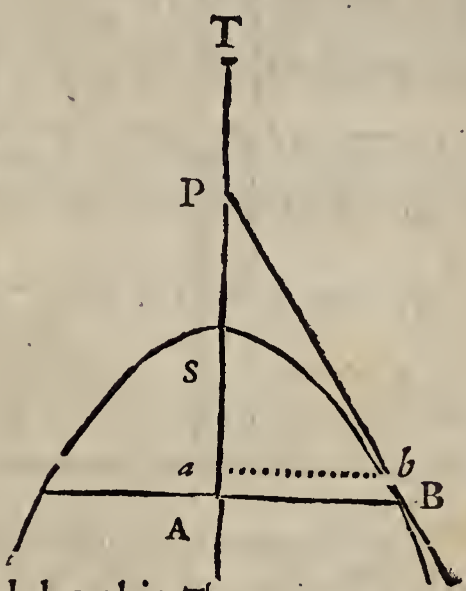
Draw any ſtrait Line at Pleaſure, in it ſet off the Length of the given Tranſverſe TS, and from its extreme Points or Limits, viz. T, S, ſet off $Tf = SF$, the Diſtance of the given Focus; (viz. the Point f without, and F within the Section, as before) that done, upon the Point f (as a Center) with any aſſum'd Radius, greater than TS, deſcribe an Arch of a Circle; then from that Radius take the Tranſverſe TS, making their Difference a ſecond Radius, with which upon the Point F within the Section, deſcribe another Arch to Cut or Croſs the firſt Arch, as at B. Then will that Point B be in the Curve of the Hyperbola, by the laſt Propoſition. And therefore 'Tis plain, that proceeding on in this Manner, you may find as many Points (like B) as may be thought convenient; (the more there are, and nearer they are together, the better) which being all joyn'd together with an even Hand (as in the Parabola) will form the Hyperbola requir'd.

There are ſeveral other Ways of delineating an Hyperbola in Plano; one Way is by finding a competent Number of Ordinates, as by Sect. 1. &c. But I think none ſo eaſy and expeditious as this Mechanical Way; I ſhall therefore, for Brevity's Sake, paſs over the reſt, and leave them to the Learner's Practice, as being eaſily deduc'd from what hath been already ſaid.

Sect.

Sect. 5. *To draw a Tangent to any given Point in the Curve of an Hyperbola.*

The drawing of a Tangent that will touch any given Point in the Curve of an Hyperbola, may be easily perform'd by Help of a Theorem, as in the Ellipsis, *Sect 6. Chap. 2.*



Let $\begin{cases} D = TS \text{ the Transverse Diameter} \\ L = \text{the Latus Rectum} \\ y = SA \text{ the Absciss}a. \end{cases}$

And $z = AP \begin{cases} \text{the Distance between the} \\ \text{Ordinate, and that Point} \\ \text{in the Transverse cut by} \\ \text{the Tangent.} \end{cases}$

Then if y be given, z may be found by this *Theorem.* $\left\{ \dfrac{Dy + yy}{\frac{1}{2}D + y} = z \right.$

Which differs from that in the Ellipsis only in the Signs, *vide* Page 371.

Or if z be given, then y may be found by this *Theorem.*

Theorem. $\sqrt{\dfrac{DD + zz}{4}} : + \frac{1}{2}D + z - \frac{1}{2}D = y$

Demonstration.

Draw the Semi-Ordinate $b\,a$, as in the Figure, and put $x = Aa \begin{cases} \text{an infinite small Space between the two Semi-Ordi-} \\ \text{nates. As before in the Ellipsis, \&c,} \end{cases}$

Then	1	$D : L :: Dy + yy : \square\, AB$
That is,	2	$TS : L :: TS + SA \times SA : \square\, AB$
1 ∴	3	$\dfrac{DyL + yyL}{D} = \square\, AB$
Again	4	$D : L :: Dy + yy - 2yx - Dx + xx : \square\, ab$
That is,	5	$TS : L :: TS + Sa \times Sa : \square\, ab$
4 ∴	6	$\dfrac{DyL + yyL - 2yxL - DxL + xxL}{D} = \square\, ab$
per *Figure*	7	$z : AB :: z - x : ab$ *Viz.* $PA : AB :: Pa : ab$
7 in □'s	8	$zz : \square\, AB :: zz - 2zx + xx : \square\, ab$
Suppose	9	$x = 0$ and every where rejected, as in the Ellipsis.
Then 3, 9	10	$zz : \dfrac{DyL + yyL}{D} :: zz - 2z : \square\, ab$
10 ∴	11	$\dfrac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} = \square\, ab$

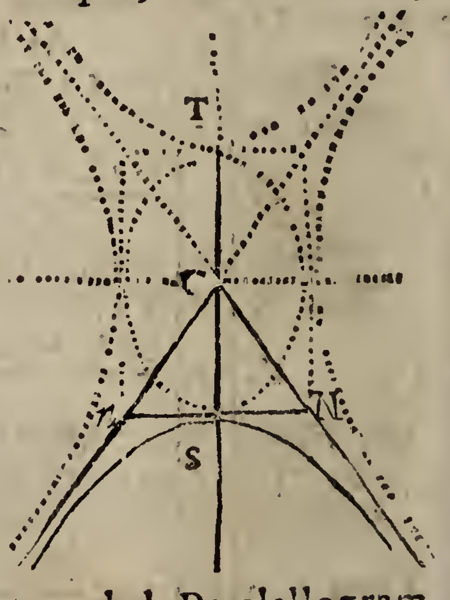
6, 11	12	$\begin{cases} \dfrac{DyL + yyL - 2yL - DL}{D} = \\ \dfrac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} \end{cases}$
12 Reduced	13	$\frac{1}{2}Dz + zy = Dy + yy$
13 *Analogy*	14	$\frac{1}{2}D + y : yy :: D + y : z.$ *Viz.* CA : SA :: TA : AP
$13 \div \frac{1}{2}D - y$	15	$z = \dfrac{Dy + yy}{\frac{1}{2}D \times y}$ Which is the first Theorem.
13 − zy	16	$yy + Dy - zy = \frac{1}{2}Dz$
16 □ C	17	$yy + Dy - zy + \dfrac{DD - 2Dz + zz}{4} = \dfrac{DD + zz}{4}$
17 ω 2	18	$y + \frac{1}{2}D - \frac{1}{2}z = \sqrt{\dfrac{DD + zz}{4}}$
18 ±	19	$y = \sqrt{\dfrac{D + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D$ Which is the *second Theorem.*

Q. E. D.

The Geometrical Effection of the first of these Theorems is very easy; for by the 14th Step it is evident, that there are three Lines given to find a fourth proportional Line. By *Problem* 3. *Page* 308.

SCHOLIUM.

From the Comparisons which have been all along made in this Chapter, between the Hyperbola and the Ellipsis, it will be easy (even for a Learner) to perceive the Coherence that is in (or between) those two Figures. But for the better understanding of what is meant by the Center and Asymptotes of an Hyperbola, consider the annex'd Scheme, wherein it is evident (even by Inspection) that the opposite Hyperbola's will always be alike, because they will always have the same Transverse Diameter common to both, *&c. See Sect.* 1. *of this Chapter.* Also, that the middle Point, or common Center of the Ellipsis, is the common Center to all the four Conjugal Hyperbola's.



And the two Diagonals of the Right-angled Paralellogram, which circumscribes the Ellipsis, or is inscrib'd to the four Hyperbola's, being continu'd, will be such Asymptotes to those Hyperbola's as are defin'd, *Chap.* 1. *Sect.* 5. *Defin.* 4.

Sect.

Sect. 6. *To draw the Asymptotes of an Hyperbola,* &c.

Having found the *Latus Rectum,* (by *Sect.* 2.) and the Conjugate Diameter n SN, in its true Position, by *Sect.* 3. Then through the Center C of the Hyperbola, and the Extreme Points n, N; of its Conjugate Diameter, draw two Right Lines, as CN, and Cn, infinitely continued, (as in the following Figure) and they will be the Asymptotes requir'd.

That is, they are two such Right Lines as, being infinitely extended, will continually incline to the Sides of the Hyperbola, but never touch them.

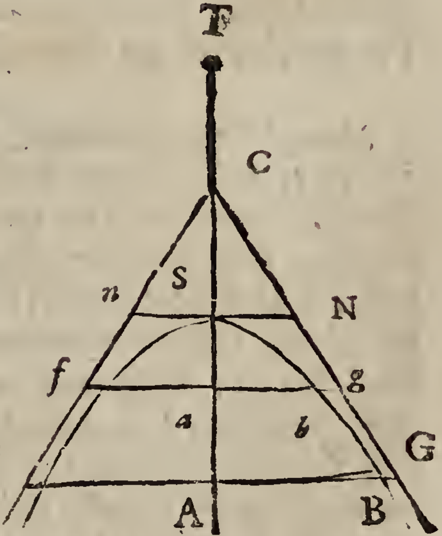
Demonstration.

Suppose the Semi-Ordinates ab and AB to be rightly apply'd to the Axis TA, and produced both Ways to the Asymptotes, as at fg and FG. Then will the $\triangle$ CSN, $\triangle$ Cag, $\triangle$ CAG, be alike,

Let $d = $ CS $=$ TC. And L $=$ the *Latus Rectum.* As before.

Put $\left\{\begin{matrix} e = Sa \\ y = SA \end{matrix}\right\}$ the Abscissa's. Then $\left\{\begin{matrix} d + e = Ca. \\ d + y = CA. \end{matrix}\right.$

Then	1	$d : SN :: d + e : ag$. *Viz.* CS : SN :: Ca : ag
1 in $\square$'s	2	$dd : \square SN :: dd + 2de + ee : \square ag$
But	3	$\frac{1}{2} d L = \square SN$. *Per* Sect. 3.
2, 3 $\therefore$	4	$\frac{ddL + 2deL + eeL}{2d} = \square ag$
Again	5	$2d : L :: 2de + ee : \square ab$. *Per* Sect. 2.
5 $\therefore$	6	$\frac{2deL + eeL}{2d} = \square ab$
4 — 6	7	$\frac{dL}{2} = \square ag - \square ab$
But $\Big\{$	8	$ag + ab = bf$ $\Big\}$ *per* Fig.
	9	$ag - ab = bg$
8 × 9	10	$\square ag - \square ab = bf \times bg$
7, 10	11	$bf \times bg = \frac{1}{2} d L$
Again	12	$dd : \square SN :: dd + 2dy + yy : \square AG$
That is.		$\square CS : \square SN :: \square CA : \square AG$
3, 12 $\therefore$	13	$\frac{ddL + 2dyL + yyL}{2d} = \square AG$
But	14	$2d : L :: 2dy + yy : \square AB$. *Per* Sect. 2.
14 $\therefore$	15	$\frac{2dyL + yyL}{2d} = \square AB$



13 — 15	15	$\frac{dL}{2} = \square AG - \square AB$
Also {	17	$AG + AB = BF$ } Per Figure.
	18	$AG - AB = BG$
17 × 18	19	$\square AG - \square AB = BF \times BG$
16, 19	20	$BF \times BG = \frac{1}{2} dL$
11, & 20 ÷	21	$bg = \frac{\frac{1}{2} dL}{bf}$. And $BG = \frac{\frac{1}{2} dL}{BF}$.

From the last Step it is evident, that the *Asymptotes* are nearer the *Hyperbola* at G, than at g, and consequently will continually approach its *Curve*. For $BF)\, \frac{1}{2} dL$ ($= BG$ is less than $bf)\, \frac{1}{2} dL$ ($= bg$, because the *Divisor* BF is greater than the *Divisor* bf; and it must needs be so, where-ever the *Ordinates* are produced to the *Asymptotes*, from the Nature of the *Triangles*.

Again, from the 7th and 16th Steps it is evident, that the *Asymptotes* can never really meet and be coincident with the *Curve* of the *Hyperbola*, although both were infinitely extended; because $\frac{1}{2} dL$ will always be the Difference between the Square of any Semi-ordinate, and the Square of that Semi-ordinate when it is produced to the *Asymptote*.

CONSECTARY.

From hence it follows, that every Right Line which passes thro' the Center, and falls within the *Asymptotes*, will cut the *Hyperbola*; and all such Lines are call'd *Diameters*, as in the *Ellipsis*, because the Properties of the *Hyperbola* and *Ellipsis* are the same.

Note, Every *Diameter*, both in the *Ellipsis*, *Parabola* and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which should they be distinctly handled, and the Effection of all such Lines as relate to them; as also, the Nature and Properties of such Figures as may be inscrib'd and circumscrib'd to all the Sections, with the various Habitude or Proportions of one *Hyperbola* to another, *&c.* would afford Matter sufficient to fill a large Volum. But thus much may suffice for an Introduction; I shall therefore desist pursuing them any farther, being fully satisfy'd, that if what I have already done be well understood the rest must needs be very easy to any one that intends to proceed farther on that Subject.

AN

AN INTRODUCTION TO THE *Mathematicks.*

PART V.

THE Method of finding out any particular *Quantity*, (viz. *either any* Line, Superficies *or* Solid) by a regular *Progression* or *Series* of *Quantities* continually approaching to it, which being *infinitely* continu'd, would then become perfectly *equal* to it, is what is commonly call'd *Arithmetick* of *Infinites*; which I shall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the *Superficial* and *Solid Contents* of *Geometrical Figures* farther on.

LEMMA I.

In any Series of equal Numbers, representing Lines or other Quantities) *As* 1. 1. 1. 1. &c. *Or* 2. 2. 2. 2. &c. *Or* 3. 3. 3. 3. &c. *If one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.*

This is so very plain and easy to be understood, that it needs no Example.

LEMMA II.

If the Series of Numbers in Arithmetical Progression begin with a Cypher, and the common Difference be 1. *As* 0. 1. 2. 3. 4. &c. (representing a Series of Lines or Roots beginning with a Point) *If the last Term be multiply'd into the Number of the Terms, the Product will be double the Sum of the Series.*

That is, putting $L =$ the last Term, $N =$ the Number of Terms, and $S =$ the Sum of all the Series.

Then

Then will $NL = 2S$. Consequently, $\frac{1}{2}NL = S$. *viz.* one half of so many Times the greatest Term, as there are Number of Terms in the Series.

Thus $\left\{ \frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} \right.$ the Sum of the Series $= \frac{1}{2}NL$. $= NL$.

And this will always be so, how many Terms soever there are, by *Consect.* 1. *Page* 185.

LEMMA III.

If a Series of Squares whose Sides or Roots are in Arithmetical Progression, beginning with a Cypher, &c. (as in the last Lemma) *be infinitely continued, the last Term being multiply'd into the Number of Terms, will be Triple to the Sum of all the Series.* viz. $NLL = 3S$, *or* $\frac{1}{3}NLL = S$.

That is, the Sum of such a Series will be one Third of the last or greatest Term, so many Times repeated as is the Number of Terms in the Series.

Instances in Square Numbers.

1. $\left\{ \frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12} \right.$

2. $\left\{ \frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18} \right.$

3. $\left\{ \frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24} \right.$ &c.

From these Instances it's evident, that as the Number of Terms in the Series does increase, the Fraction or Excess above $\frac{1}{3}$ does decrease, the said Excess always being $\frac{1}{6N-6}$ which, if we suppose the Series to be infinitely continued, will then become infinitely small, *viz.* in Effect nothing at all.

Consequently, $\frac{1}{3}NLL$ may be taken for the true or perfect Sum of such an infinite Series of Squares.

LEMMA IV.

If a Series of Cubes, whose Roots are in Arithmetical Progression, beginning with a Cypher, &c. (as above) *be infinitely continued, the Sum of all the Series will be* $\frac{1}{4}NLLL = S$.

That is, is one Fourth of the last or greatest Term, so many Times repeated as is the Number of Terms.

Instances

Instances in Cube Numbers.

If o . 1 . 2 . 3 . *&c.* be the Roots of the Cubes.

Then 1. $\left\{\frac{0+1+8+27}{27+27+27+27}=\frac{36}{108}=\frac{4}{12}=\frac{1}{4}+\frac{1}{12}\right.$

2. $\left\{\frac{0+1+8+27+64}{64+64+64+64+64}=\frac{100}{320}=\frac{10}{32}=\frac{5}{16}=\frac{1}{4}+\frac{1}{16}\right.$

3. $\left\{\begin{array}{l}\frac{0+1+8+27+64+125}{125+125+125+125+125+125}=\frac{225}{750}=\frac{45}{150}=\frac{3}{10}\\ =\frac{6}{20}=\frac{1}{4}+\frac{1}{20}\end{array}\right.$

From all these Examples it plainly appears, that as the Number of Terms in the Series increases, the Fraction or Excess above $\frac{1}{4}$ decreases; the Excess being always $\frac{1}{4N-4}$ which, if we suppose the Series to be infinitely continued, will become infinitely small, or rather nothing. As in the last Lemma.

Consequently, $\frac{1}{4}$ NLLL may be taken for the true and perfect Sum of all the Terms in such an infinite Series of *Cubes.*

LEMMA V.

If a Series of Biquadrates, whose Roots are in Arithmetical Progression, beginning with a Cypher, &c. as before, be infinitely continued, the Sum of all the Terms in such a Series will be $\frac{1}{5}$ NL^4.

The Truth of this may be manifested by the like Process, as in the foregoing Lemma's, and so on for higher Powers. But if any one desires a farther Demonstration of these Series, he may (I presume) meet with ample Satisfaction in Dr. *Wallis*'s History of *Algebra,* Chap. 78 and 79, wherein the Doctor concludes with these Words:

" Thus having shew'd, that in a Progression of Laterals (or A-
" rithmetical Proportionals) beginning at o. the Sum of 2, 3, 4,
" 5, 6 Terms, is always equal to $\frac{1}{2}$ of so many times the greatest,
" and there being no Pretence of Reason why we should then
" doubt it, in a Progression of 7, 8, 9, 10, *&c.* we conclude it so to
" be, tho' such a Number of Terms be supposed infinite.

" Again, in a Progression of their Squares, having shew'd, that
" in 2, 3, 4, 5, 6 Terms, the Aggregate is always more than $\frac{1}{3}$ of so
" many times the greatest; and the Excess always such aliquot Part

" of

" of the greateſt, as is denominated by ſix times the Number of
" Terms wanting 1. (As if the Terms be 2, it is $\frac{1}{3}+\frac{1}{6}$; if 3, it
" is $\frac{1}{3}+\frac{1}{12}$; if 4, it is $\frac{1}{3}+\frac{1}{18}$; if 5, it is $\frac{1}{3}+\frac{1}{24}$ of ſo many times the
" greateſt Term, and ſo onward) we may well conclude, (there
" being no Pretence of Reaſon why to doubt it in the reſt) that
" it will be ſo, how many ſoever be ſuch Number of Terms.
" And becauſe ſuch Exceſs as the Number of Terms do increaſe,
" will become infinitely ſmall (or leſs than any aſſignable) we con-
" clude (from the Method of Exhauſtions) that, if the Number of
" Terms be ſuppoſed infinite, ſuch Exceſs muſt be ſuppoſed to va-
" niſh, and the Aggregate of ſuch infinite Progreſſion ſuppoſed
" equal to $\frac{1}{3}$ of ſo many times the greateſt.

" In like manner, having prov'd that ſuch Progreſſion of Cubes
" doth (as the Number of Terms encreaſeth) approach infinitely
" near to $\frac{1}{4}$ of ſo many times the greateſt) and of Biquadrates to $\frac{1}{5}$,
" and ſo of Surſolids to $\frac{1}{6}$ of ſo many times the greateſt, and ſo
" onwards as we pleaſe to try, and there being no Pretence of Rea-
" ſon why to doubt it as to the reſt) we may take it as a ſufficient
" Diſcovery, that (univerſally) the Aggregate of ſuch infinite Pro-
" greſſion is equal (or doth approach infinitely near to ſuch a Part
" of ſo many times the greateſt, as is denominated by the Exponent
" or Number of Dimenſions) of ſuch Power (as is that according
" to which the Progreſſion is made) increaſed by 1. namely, of La-
" terals $\frac{1}{2}$; of Squares $\frac{1}{3}$; of Cubes $\frac{1}{4}$; of Biquadrates $\frac{1}{5}$; (of ſo ma-
" times the greateſt) and ſo onwards infinitely.

This Diſcourſe of the Doctor's I thought convenient to inſert ſuppoſing it may give ſome Satisfaction to the Learner, to hear ſo great a Man as Dr. *Wallis*'s Arguments about the Truth of theſe Series, which I have briefly deliver'd in the foregoing Lemma's.

LEMMA VI.

If any two Series or Ranks of Proportionals, have the ſame Number of Terms, (whether Finite or Infinite) *it will always*

be { *As the firſt Term of one Series : is to the firſt Term of the other Series : : ſo is the Sum of all the Terms in the one Series : to the Sum of all the Terms in the other Series.* (12. *c.* 5.)

As

As in these Numbers,

1	3
2	6
3	9
4	12
5	15
6	18

Or these Numbers

4	5
12	15
36	45
108	135
324	405
972	1215

That is, 1 : 3 :: 21 : 63   And 4 : 5 :: 1456 : 1820 &c.

The Application of these Lemma's to Geometrical Quantities, *viz.* to Lines, Superficies and Solids, wholly depends upon granting the following Hypotheses.

The Hypotheses.

1. That every Line is suppos'd to consist or be compos'd of an infinite Series of equidistant Points.

2. A Surface (*viz.* the Area of any Figure) to consist of an infinite Series of Lines, either strait or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Planes, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficicies; or that Planes, which have not any Thickness, can constitute a Solid: But by what we here call Lines, are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so, as that their Breadths, being all taken and put together, must be equal to the Figure they are suppos'd to fill up.

And those Planes or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so, as that their Depths or Thicknesses (which are hereafter also call'd Lines) being all taken together, must be equal to the Height of tht propos'd Solid.

Now, in order to render these Hypotheses as easy for a Learner to understand as I can, I shall here propose a very plain and familiar Example.

Viz. Let us suppose any Book to be compos'd or made up of 100, 200. 300 (more or less) Leaves of fine Paper; such a Book, being close put together, will have Length, Breadth and Depth or Thickness, and therefore may not (improperly) be call'd a Solid; and each of its Edges (being evenly cut) will be a Superficies compos'd of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves

 to

to be divided into 10, or 100, or 1000, &c. they will then become such a Series of infinitely small Lines, as are (by the Hypothesis) said to compose, or fill up a Superficies.

And all the Superficies of those infinitely thin, or divided Leaves of Paper, will become such a Series of Planes, or Superficies, as are said to constitute a Solid, *viz.* such a Solid as the Bigness and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the least Prejudice to any Demonstration, admit of the following Definitions, and Theorems.

Definitions.

I. The Area's of Squares, and all other Parallelograms, are compos'd or fill'd up with an infinite Series of equal Right-lines.

II. The Area of every plane Triangle, is compos'd of an infinite Series of Right-lines parallel to its Base, and equally decreasing, until they terminate in a Point at the Vertical Angle.

III. The Area of a Circle may be compos'd either of an infinite Series of concentric or parallel Circles; or of an infinite Series of Chord Lines parallel to its Diameter; or of an innumerable Multitude of Sectors.

IV. The Area of an Ellipsis may be compos'd, either of an Infinite Series of Ordinates rightly apply'd, or of an infinite Series of Right-lines parallel to its Transverse-Diameter.

V. The Area's of the Parabola, and Hyperbola, are compos'd of an infinite Series of Ordinates, or may also be compos'd of Right-lines parallel to their Axis, &c.

VI. A Prism is a solid Body, contain'd or included within several equal Parallelograms, having its Bases or Ends equal and alike; and 'tis generally named according to the Figure of its Base. That is,

VII. A Cube (or Solid like a Dye) is a Prism bounded or included within six equal Square-Planes.

VIII. A Parallelopipedon is a Prism that hath its Sides bounded or included within four equal Parallelograms, and two Square-Bases or Ends.

IX. A Cylinder (or Solid like the Rolling-Stone in a Garden) is only a round Prism, having its Bases or Ends a perfect Circle.

X. The

X. The Solidity of every Prifm is compos'd of an infinite Series of equal Planes, parallel and alike to that of its Bafe.

XI. A Pyramid is a Solid bounded or included within feveral plane Triangles, fet upon any Polygonous Bafe, having their Vertical-Angles all meeting together in a Point, call'd the Vertex, and takes its Name from the Figure of its Bafe, *viz.* if it have a a Square-Bafe, 'tis called a Square-Pyramid, if a Triangle-Bafe, 'tis call'd a Triangular-Pyramid, *&c.*

XII. A Cone is always a round Pyramid, which hath been already defin'd in *Page* 361, *&c.*

XIII. The Solidity of every Pyramid is compos'd or conftituted of an infinite Series of Planes, parallel and alike to that of its Bafe, equally decreafing until they terminate in a Point at the Vertex.

XIV. A Sphere or Globe (*viz.* a Ball) is a Solid bounded or included within one regular Superficies, being form'd or generated by the Rotation of a Semi-circle about its Diameter; (call'd the Axis of the Sphere) and its Solidity is compos'd or conftituted of an infinite Series of concentric Circles, whofe Diameters are the Chords of that Circle by which it was form'd.

XV. A Spheroid (or Egg-like Figure) is a Solid bounded with one regular Superficies. form'd by the Rotation of a Semi-Ellipfis about its Tranfverfe-Diameter (call'd the Axis of the Spheroid) and its Solidity is conftituted of an infinite Series of concentric Circles, whofe Diameters are the Ordinates of that Ellipfis by which it was form'd.

XVI. There is another Sort of a Solid call'd an Oblate-Spheroid, being form'd by the Rotation of an Ellipfis about its Conjugate-Diameter; and 'tis like to a flat Turnip.

XVII. If a Semi-parabola be turn'd about its Axis, it will form a Solid call'd a Parabolic-Conoid, being compos'd or conftituted of an infinite Series of Circles, whofe Diameters are the Ordinates of the Parabola.

XVIII. If a Parabola be turn'd about its Bafe or greateft Ordinate, it will form a Solid call'd a Pyramidoid, but moft commonly a Parabolic Spindle, which will be conftituted of an infinite Series of Circles, whofe Diameters are Right-lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turn'd about its Axis, it will form a Solid call'd an Hyperbolic-Conoid, being conftituted of an infinite Series of Circles, whofe Diameters are the Ordinates of the Hyperbola.

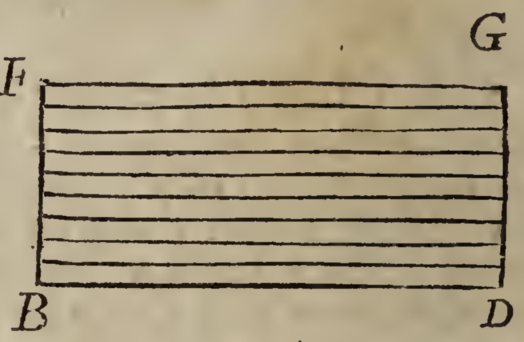
XX. The curve Superficies of all circular Solids (*viz.* Cylinders, Cones, Spheres, *&c.*) are compos'd of an infinite Series of the Peripheries of those Circles which constitute their Solidities.

Upon these Definitions are grounded all the following Theorems; and therefore, if they were diligently compar'd with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very easy; wherein I shall begin with what hath been already demonstrated, by way of introducing the rest.

THEOREM I.

The Area of every Right-lin'd Parallelogram, is obtain'd by multiplying the Length into its Breadth.

That is, $BD \times FB =$ the Area of the Parallelogram, $BDFG$. By Lemma 1. compar'd with Definition 1.



Example, Suppose $BD = 26$. And $FB = 9$.

Then $26 \times 9 = 234$ the Area. See Problem 1. *Page* 339.

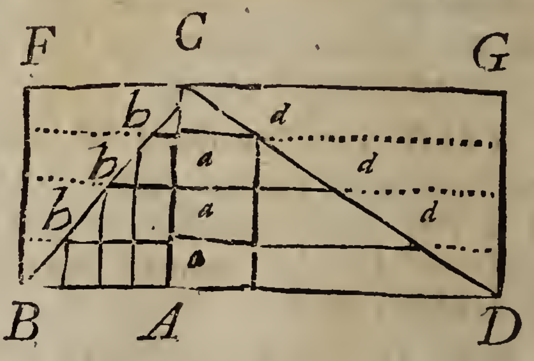
THEOREM II.

The Area of every plane-Triangle is equal to Half the Area of its circumscribing Parallelogram.

That is, $\frac{BD \times CA}{2} =$ the Area of $\triangle BCD$, in the following Figure.

Demonstration.

Suppose the Perpendicular CA to be divided into an infinite Number of equal Parts, as at the Points *a, a, a,* &c. and through those Points there were drawn Right-lines parallel to the Base *BD, viz. b, a d, b a, d, b, a, d,* &c. Then will those Lines be a Series of Terms in Arithmetical Progression, beginning at the Point C. *viz.* 0, *bd*, 2,*bd*, 3,*bd*, &c. as is evident by the Figure, wherein BD is the greatest Term $=$ L, and CA the Number of Terms $=$ N.



But

But $\frac{1}{2}$ N L = S, by Lemma 2. And S = the Triangle's Area, by Definition 2. Q. E. D.

Example, Let B D = 26, and C A = 9. As above.

Then $\frac{26 \times 9}{2} = 117$. Or $\frac{26}{2} \times 9 = 117$. Or thus $26 \times \frac{9}{2} = 117$. the Area requir'd. See Problem 3. *Page* 330.

THEOREM III.

The Peripheries of Circles are in Proportion one to another, as their Diameters are.

Demonstration

Let the Periphery of a Circle be divided into any Number of equal Arches, by Right-lines drawn from the Center, *viz.* Radius's) suppose them 8, as in the annex'd Figure, wherein A B is one of them. Then if through any Point in the Radius there be drawn a concentric or parallel Circle, its Periphery will also be divided into 8 equal Arches by those Radius's; one whereof will be *a b*, and the △ C *a b* will be like to △ C A B.

Therefore C *a* : *ab* :: C A : A B. Or C *a* : C A :: *ab* : A B

Consequently 2 C *a* : 2 C A :: 8 *a b* : 8 A B.

But 2 C *a* = *d a* the Diameter of the Circle, whose Periphery is 8 *a b*.

And 2 C A = D A the Diameter of the Circle, whose Periphery is 8 A B. Therefore, *&c.* as by Theorem. Q. E. D.

Example. In *Chapter* 6. *Part* III. it was found, that if the Diameter of a Circle be 2. its Periphery will be 6,2831853071795864

Ergo 2 : 6,2831853, &c. : : 1 : 3,14159265, &c. the Periphery of the Circle whose Diameter is 1.

Corollary.

Hence it follows, that because Unity or 1. may be made the first Term in the Proportion, therefore 3,14159265, &c. may be made a constant or settled Factor; which being multiply'd into any propos'd Diameter, will produce the Periphery of that Circle.

Note. Instead of 3,14159265, &c. it may be sufficient to take only 3,1416.

Or

Or in whole Numbers the Proportion may be.

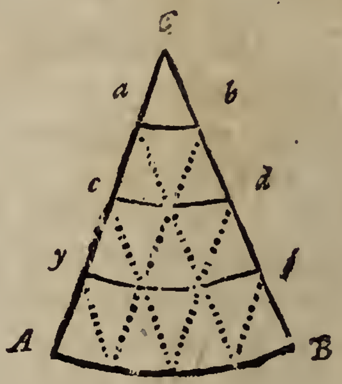
As 7 : 22 :: Diam :: Periphery } { these Numb. may serve, and
Or 113 : 355 :: Diam :: Periphery } { are often us'd in com. Practice.

THEOREM IV.

The Area of any Sector of a Circle, is equal to Half the Rectangle of the Radius into its Arch.

That is, $\frac{CA \times AB}{2} =$ the Area af ACB.

Demonstration.



Suppose the Radius CA to be divided into an infinite Series of equidistant Points, as *a*, *e*, *y*, *&c.* and through those Points there were drawn concentric or parallel Arches, as *ab*, *ed*, *yf* *&c.* Then they will be a Series of Arches in Arithmetical Progression, beginning at the Point C, (*viz.* 0, 1, 2, 3, *&c.*) as plainly appears by the Figure, wherein the greatest Term is $AB = L$, and Number of Terms is $CA = N$. But $\frac{1}{2} NL = S$ the Sum of all the Series, by Lemma 2. And $S =$ the Sector's Area, by Definition 3.

Q. E. D.

Example. Let the Radius $CA = 12$. And the Arch $AB = 8$. Then $\frac{12 \times 8}{2} = 48$. Or $\frac{12}{2} \times 8 = 48$. Or $\frac{8}{2} \times 12 = 48$. the Area of the Sector ACB.

THEOREM V.

The Area of every Circle, is equal to Half the Rectangle of the Radius into its Periphery.

That is, according to Archimedes, *a Circle is equal to a right-angled Triangle, whose Sides containing the Right-angle, are equal, one to the Radius, and the other to the Perimeter of that Circle.* Pro 1. de dimensione circuli.

The Truth of this Theorem may be easily deduc'd from the last, thus, if we suppose the last Sector to be one eighth Part of a Circle; then it follows, that $\frac{8 AB \times CA}{2} = 4 AB \times CA$ will be the Area of the whole Circle.

But $4 AB =$ Half the Circle's Periphery; and $CA =$ Half its Diameter. Therefore, *&c.* As *per* Theorem. Q. E. D.

Example

Example, If the Diameter be Unity, or 1. the Periphery will be 3,14159265, *&c.* by Theorem 3.

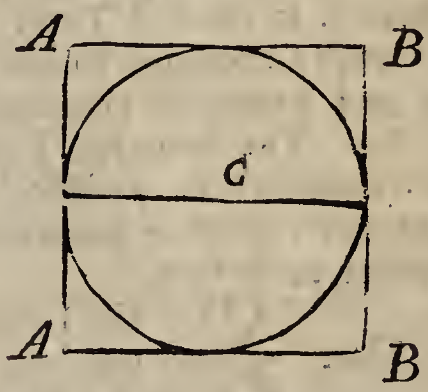
Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$, *&c.* (Or 0,7854 for common Use) will be the Area of that Circle.

SCHOLIUM.

From hence naturally flows the following Proportion between the Square, and its inscrib'd Circle.

Proportion. *As the Perimeter* (viz. *the Sum of the four Sides*) *of any Square : is to its Area :: So is the Periphery of the inscrib'd Circle : To its Area.*

That is, supposing A B = D = the Side of the Square, and the Diameter of its inscrib'd Circle.



Then 4 D = the Perimeter ; D D = the Area of the Square; and 3,1416 D = the Periphery of the Circle. By Theorem 3.

But 4D : D D :: 3,1416 D : 0,7844DD = the Circle's Area.

And if D = 1. Then 4D = 4 and D D = 1 × 1 = 1. And the Periphery will be 3,1416,

Then 4 : 1 :: 1 : 0.7854 *&c.* As in the Example above.

And from hence may be easily deduc'd the following Theorem.

THEOREM VI.

The Area's of all Circles are in Proportion one to another, as the Squares of their Diameters. (2. *e.* 12.)

For if D = the Diameter of one Circle, and *d* = the Diameter of another Circle.

Then will 0.7854DD be the Area of one Circle, and 0,7854*dd* will be the Area of the other Circle, as above.

But 0,7854 D D : 0,7854*dd* : : D D : *d d*. Or thus.

Let D = the Diameter, and P = the Periphery of one Circle: *d* = the Diameter, and *p* = the Periphery of another Circle.

Then	1	$\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$ the Area of one Circle.
And	2	$\frac{1}{2}d \times \frac{1}{2}p = \frac{1}{4}dp = a$ the Area of the other Circle.
1 × 4	3	$DP = 4A$ (*per* last Theorem.
2 × 4	4	$dp = 4a$
3 ÷ D	5	$P = \frac{4A}{D}$

4 ÷ *d*

4 $\div$ d	6	$p = \frac{4a}{d}$
But	7	P : p : : D : d. *Per* Theorem 3.
5, 6, 7	8	D : d : : $\frac{4A}{D}$: $\frac{4a}{d}$
8 $\because$	9	$4DDa = 4ddA$. That is, $DDa = ddA$
9, *Anal.*	10	D D : A : : dd : a. Or A : a : : D D : dd.

Q. E. D.

Corollary.

Hence it follows that becauſe the Square of 1 is 1. (*viz.* $1 \times 1 = 1$) and 0,785398 16, *&c.* Or 0,7854 is the Area of the Circle whoſe Diameter is 1. (as before) therefore it will be 1 : 0,7854 : . So is the Square of any Circle's Diameter : To its Area. And becauſe 1. is the firſt Term in the Proportion, therefore 0,7854 may be made a conſtant Factor; which being multiply'd into the Square of any propos'd Diameter, will produce the Area of that Circle.

Note, The four laſt Theorems do plainly ſhew the Reaſon of all the common or practical Problems about a Circle; which for the Learner's further Satisfaction, I have here inſerted together; ſuppoſing as before,

That $\left\{ \begin{array}{l} D = \text{the Diameter} \\ P = \text{the Periphery} \\ A = \text{the Area} \end{array} \right\}$ of any propos'd Circle.

Then		*Prob.* 1. D being given, to find P.
	1	1 : 3,1416 : : D : P. *Per* Theorem 3.
1 $\because$	2	3,1416 D = P
Example.		Suppoſe D = 32. Then $3,1416 \times 32 = 100,5312$ the Periphery.
		Prob. 2. D, being given, To find A.
	3	1 : 0,7854 : : D D : A. *Per* Theorem 6.
3 $\because$	4	0,7854 D D = A
Example.		Suppoſe D = 32. As before.
Then		$DD = 32 \times 32 = 1024$.
And		$0.7854 \times 1024 = 804.2496$ the Area requir'd.
		Prob. 3. P, being given, to find D.
2 $\div$	5	$D = \frac{P}{3,1416}$ Or $\left\{ \begin{array}{l} \text{becauſe } \frac{1}{3,1416} = 0,3183 \\ \text{therefore } 0.3183\ P = D. \end{array} \right.$
		This being only converſe to the firſt needs no *Exam.*

		Prob. 4. P, being given, To find A.
2 ⊕ 2	6	$9,86965\ DD = PP$
6 ÷	7	$DD = \frac{PP}{9,86965}$ Or $0,10132\ PP = DD$
4 ÷	8	$DD = \frac{A}{0,7854}$ Or $1,2732\ A = DD$
For		$\frac{1}{0,7854} = 1,2732$
7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$ Or $0,10132\ PP = 1,2732\ A$
9 × &c.	10	$\frac{PP}{12,5664} = A$ Or $0,07957\ PP = A$
		Prob. 5. A, being given, To find D.
8 ɯ 2	11	$D = \sqrt{\frac{A}{0,7854}}$ Or $D = \sqrt{1,2732\ A}$
		Prob. 6. A, being given, To find P.
10 × &c.	12	$PP = 12,5664\ A$ Or $PP = \frac{A}{0,07957}$
12 ɯ 2	13	$P = \sqrt{12,5664\ A}$ Or $P = \sqrt{\frac{A}{0,07957}}$

These six Problems contain all the Variety that can be proposed about finding the Periphery, Diameter, and Area of any Circle.

But if it be requir'd to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Consideration.

First, As to the Data, there must always be given the Diameter; or, either the Periphery, or Area of the Circle, in order to find the Diameter.

Secondly, There must also be given, either the Chord, which is the Base of the Segment, or the Versed Sine, which is the Height of the Segment.

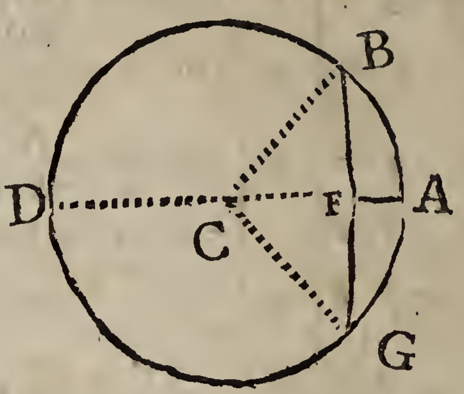
That is, either BG, or AF, in the following Scheme, must be given, that so the Area of the △ BCG may be found.

Then it's evident, (by the Figure) that if the Area of the △ BCG be taken from the Area of the Sector CBAG, the Remainder will be the Area of the Segment BAG.

And if the Area of the Segment BAG be taken from the whole Area of the Circle, the Remainder will be the Area of the other Segment DBG.

Example in *Numbers*.

Let there be given DA = 32. as *Prob.* 1.
And the Versed Sine AF = 6.
Then $\frac{1}{2}$DA = BC = CA = 16.
And CA — AF = CF = 10.
But □ BC — □ CF = □ BF.
Consequently $\sqrt{\square BC - \square CF} = BF$.
Viz. $\sqrt{156} = 12,49 = BF$.



Then by the Doctrine of plane Triangles, the Arch BA = ∠ BCA may be found in Degrees and Decimal Parts;
Thus BC : *Radius* :: BF : *Sine* ∠ BCF = 51°, 31′
And then it will always hold in this Proportion;

Viz. { *As the Circle's Periphery in Degrees* : *is to its Periphery in equal Parts.* (according to the Dimensions taken) : : *So is the Arch in Degrees* (viz. ∠*BCA*) : *To the same Arch in equal Parts.*

That is, 360° : 100,5312 :: 51°,31′ : 14,3284 = BA
Then 14,3284 × 16 = 229,2544 the Area of the Sector BCGA.
And 12,49 × 10 = 124,9 the Area of the △ BCG.
Their Difference 104.3544 = the Area of the Segm. BAG.

Or the Area of any Segment may be otherwise found (as most usually it is) by a Table of the Segments of a Circle, whose Area is Unity, or 1. The Construction or making of such a Table is very well laid down in Mr. *Darie*'s Book of Gaging, *Chap.* 9. which he performs in this *Problem.*

PROBLEM

In a Circle whose Area is Unity, and its Diameter cut by Chord Lines into 1000 *equal Parts, to find the Segment to any Versed Sine propos'd, not exceeding* 500 *of those equal Parts.*

1. Multiply the Versed Sine propos'd by 0,002, and subtract the Product from an Unit or 1.

2. This Remainder you shall seek in the Common Table of Natural Sines, (the Arch being divided into Degrees and Centesima's) which being found, let its Co-arch be doubled, and call'd A.

3. You must find the Correspondent Sine to A; which Sine being found, you may call S, and then it holds
6,2831853) 0,0174532925 A — S (= the Segment requir'd.

Now

Now this Segment being thus found, if you ſubduct it from an Unit, you have the Co-ſegment, *&c.*

Note, Notwithſtanding what hath been ſaid in the ſecond Precept of this Problem, it very often falls out, that the Remainder there ſpoken of, cannot be truly found in the Table of Natural Sines; therefore in this Caſe my Advice is, that you make two Operations, one with a Sine the next Greater, and one with a Sine the next Leſs; and in ſo doing you will be ſure to have the Segment requir'd bounded between the Reſults of thoſe two Operations.

Example, *Let it be propos'd to find the Correſpondent Segment to the Verſed Sine* 263.

Firſt, 263 × 0.002 = 0,526, and 1 — 0,526 = 0,474, its Arch is 28°.29 being leſs than juſt; its Complement is 61,71 which being doubled, is 123,42 = A,

Then ,0174533 A = 2,154086286
— 0,8346556 = S The Sine of A.

6,2831853) 1,319430686 (0,209993 The Segment.

Now I make a ſecond Work.

263 being multiply'd with 0,002 is 526. and 1 — 526 = 0,474 its Arch is 28°,30′ being greater than juſt; and its Complement is 61,70, which being doubled is 123,4 = A.

Then 0,0174533 A = 2,1537372
— 0,8348478 = S The Sine of A.

6,2831853) 1,3188894 (0,209907 The Segment.

So you ſee by theſe two Operations, that the Segment is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this large Factor, and this large Diviſor, I ſhall here inſert two Tablets of them, which will be ready for Uſe, and exact enough too.

Diviſor.	
6,2832	1
12,5664	2
18,8495	3
25,1327	4
31,4159	5
37,6991	6
43,9823	7
50,2655	8
56,5487	9

Factor.	
,0174533	1
,0349066	2
,0523599	3
,0698132	4
,0872665	5
,1047197	6
,1221730	7
,1396263	8
,1570796	9

Thus far Mr. *Darie,* which I have here inſerted to ſhew the Learner how, by the Help of theſe two Tablets, and a Table of Natural Sines, he may eaſily make a Table of Segments, whoſe Uſe ſhall be ſhew'd farther on, *viz.* when I come to Practical Gaging. In the mean Time I ſhall here lay down anothe Method, to find the Area of any Segment of a Circle (very near) by a

 new

new *Theorem*, without the Help either of a Table of Sines or Segments, having the [illegible] Data as before in *Page* 404.

Viz. Let $\begin{cases} R = \text{the Radius, or } \frac{1}{2} \text{ Diameter of the given Circle.} \\ d = \text{the Difference between the Versed Sine and Radius.} \\ C = \text{half the Chord of the Segment's Base.} \end{cases}$

Theorem $\left\{ \dfrac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{3}R + d} \right. \times C = S$, the Area of the Segm.

Example, Suppose $R = BC = 16$. $d = FC = 10$. and $C = BF = 12{,}49$. As before.

Then $2\frac{1}{3}RR = 597{,}3333$. $1\frac{1}{3}Rd = 213{,}3333$. $dd = 100$
$-313{,}3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34)\ 284{,}0000\ (8{,}3529$

Lastly, $8{,}3529 \times 12{,}49 = 104{,}3276$ the Area of the Segment *BAG*, As before.

THEOREM VII.

As Squares are to the Area's *of their inscrib'd Circles, so are Parallelograms to the* Area's *of their inscrib'd Ellipses.*

That is, $\begin{cases} \text{\textit{As the Square of the Diameter of any Circle : is}} \\ \text{\textit{to its} Area \textit{:: so is the Rectangle of the Transverse}} \\ \text{\textit{and conjugate Diameters of any Ellipsis to its} Area.} \end{cases}$

Demonstration.

Circumscribe any Ellipsis with a Circle; and suppose an infinite Number of Chord-Lines drawn therein, all parallel to the Conjugate Diameter; as those in the annex'd Figure; then it will be $\begin{cases} \text{\textit{As (DA) the Diameter of the Circle : is to (Nn) the conjugate Diameter of}} \\ \text{\textit{the Ellipsis :: So is (BaB) any Chord in the Circle : to (bab) its respective}} \\ \text{\textit{Ordinate in the Ellipsis.}} \end{cases}$

For according to the Property of the Circle

it is	1	$\overline{TS - Ta} \times Ta = \square\, Ba$
And		by the Property of the Ellipsis
it is	2	$\square TC : \square NC :: \overline{TS - Ta} \times Ta : \square ba$
1, 2	3	$\square TC : \square NC :: \square Ba : \square ba$
3, hence	4	$TC : NC :: Ba : ba$
Conseq.	5	$2TC : 2NC :: 2Ba : 2ba$
That is,	6	$DA : Nn :: BaB : bab$
Put	7	$D = 2TC$, and $d = 2NC$
Then	8	$D : d ::$ *Chord* BaB: *Ordinate* bab, &c.

But

But the Sum of an Infinite Series of ſuch Chords, as *BaB*, do conſtitute the Area of the Circle, by Definition 3.

And the Sum of the like Series of their reſpective Ordinates, as *bab*, do conſtitute the Ellipſis's Area, by Definition 4.

Therefore *D* : *d* :: Circle's Area : Ellipſis's Area, by *Lemma* 6.

But *D* : *d* :: *DD* : *Dd*. Whence it follows,

That *DD* : Circle's Area :: *Dd* : Ellipſis's Area. Q. E. D.

Conſequently, As 1 : is to 0,7854 : : ſo is the Rectangle, or Product of the Tranſverſe, and Conjugate Diameters of any Ellipſis : To its Area.

Example, Suppoſe $TS = 36$. and $Nn = 16$. Then $36 \times 16 = 576$. And $576 \times 0,7854 = 452,3904$ the Area of the Ellipſis.

Corollaries.

1. Hence it is eaſy to conceive, that the Square Root of the Rectangle or Product of the Tranſverſe, and Conjugate Diameters, will be the Diameter of a Circle, whoſe Area will be equal to the Ellipſis's Area.

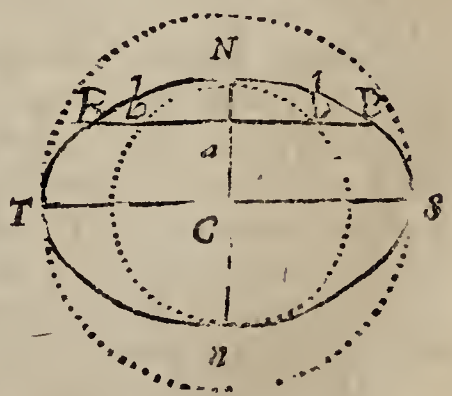
Viz. $\sqrt{576} = 24$ the Diameter of a Circle = to the Ellipſis.

2. All Segments of an Ellipſis and its circumſcribing Circle, (whoſe Baſes are parallel to the conjugate Diameter, and of the ſame Height) are in Proportion one to another, as their Baſes are. That is, *BaB* : *bab* : : Area Segment *BTB* : Area Segment *bTb*; Or *TS* : *Nn* :: Area Segment *BTB* : Area Segment *bTb*.

THEOREM VIII.

The Area *of every Ellipſis, is a mean Proportional between the* Area's *of its Circumſcribing, and inſcrib'd Circles.*

The Truth of this *Theorem* may be eaſily deduc'd from the Laſt; for ſuppoſing $D = TS$. and $d = Nn$, as before. Then it is already prov'd, that *DD* : *Dd* : : Circumſcribing Circle's Area : Ellipſis's Area.



But *DD* : *Dd* : : *Dd* : *dd*.

Therefore, Ellipſis Area : Inſcrib'd Circle's Area : : *Dd* : *dd*.

By *Theorem* 6.

Example, Let $TS = D = 36$. and $Nn = d = 16$. As before.

Then $DD = 1296$. and $dd = 256$.

Then

Then will $\begin{cases} 1296 \times 0{,}7854 = 1017{,}8784 \text{ the great Circle's Area} \\ 256 \times 0{,}7854 = 201{,}0624 \text{ the lesser Circle's Area.} \end{cases}$

Suppose $A =$ the Ellipsis Area. Then, according to the *Theorem*, it will be, $1017{,}8784 : A :: A : 201{,}0624$.

Ergo $AA = 1017{,}8784 \times 201{,}0624 = 204657{,}07401216$.

Consequently, $\sqrt{204657{,}07401216} = 452{,}3904 = A$, the Area of the Ellipsis, As before in the last *Example*.

Corollary.

From hence it follows, that all Segments of an Ellipsis and its inscrib'd Circle, whose Bases are parallel to the Transverse Diameter, and have the same Height, are in Proportion one to another as the Area's of the Ellipsis and Circle are.

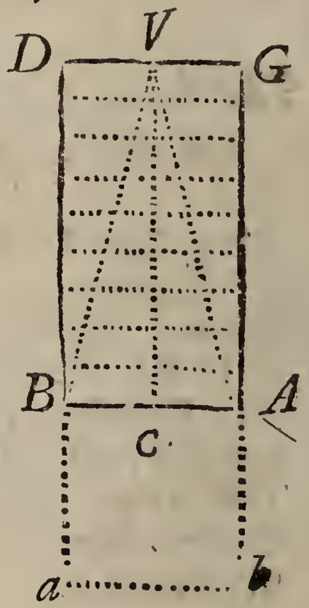
That is, Area of Circle : Area of Ellipsis :: Segment bNb : Segment BNB.

Or, $Nn : TS ::$ Area Segment bNb : Area Segment BNB.

THEOREM IX.

The Solid Content of any Prism (what Figure soever its Base is of) *is obtain'd by multiplying the* Area *of its Base into its Height.*

For Instance, a Parallelopipedon (or square Prism) is constituted of an infinite Series of equal Squares; that of its Base $BAba$ being one of the Terms, and its Height DB, or GA the Number of all the Terms.



Consequently, the Area of $BAba \times DB =$ the Sum of all the Series (by *Lemma* 1.) which is the Solidity of the Parallelopipedon $DBGA$, by Definition 10.

Example, Suppose the Side of the Base $BA = 16$ and the Height $DB = 42$.

Then will $16 \times 16 = 256$ be the Area of the Base.

And $256 \times 42 = 10752$ the solid Content of the Parallelopipedon $DBGA$.

In this Manner you may find the Solidity of all regular Polygonous Prisms, whose Bases (or Ends) are parallel and alike, what Form soever they are of.

That is, whether their Bases are Triangles, Pentagons, Hexagons, or Octagons, &c.

THEOREM

THEOREM X.

Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it. (7. e. 12.)

That is, the *solid Content* of the *Pyramid BVA*, (*in the last Figure*) is one *Third* of *its circumscribing Prism DBGA.*

Demonstration.

For every Pyramid that hath a square Base (as *BAba*, in the last Figure) is constituted of an infinite Series of Squares, whose Sides or Roots are continually increasing in Arithmetical Progression, beginning at the Vertex or Point *V* (See *Theor.* 2.) its Base *BAba*, being the greatest Term, ($= LL$) and its perpendicular Height *VC*, or *DB*, is the Number of all the Terms ($= N$); but $\frac{NLL}{3} = S$ the Sum of all the Series, by *Lemma* 3. and $S =$ the solid Content of the Pyramid *BVA*, by Definition 13.

Example, Suppose the Side of a Pyramid's Base be $BA = 16$. and its Height be $VC = 42$. Then $16 \times 16 = 256$ the Area of its Base *BAba*. And $\frac{256 \times 42}{3} = 3584$. Or $\frac{256}{3} \times 42 = 3584$.

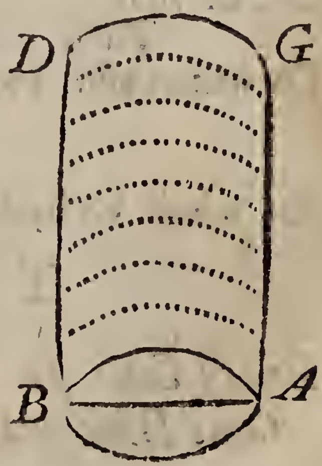
Or thus, $256 \times \frac{42}{3} = 3584$, is the Solidity of that Pyramid *BVA*.

Corollary.

From hence it will be easy to conceive, that every Pyramid is $\frac{1}{3}$ of its circumscribing Prism, what Form soever its Base is of, *viz.* whether it be a Square, Triangle, or Pentagon, &c.

THEOREM XI.

The solid Content of every Cylinder, is obtain'd by multiplying the Area *of its Base into its Height.*



For every Right Cylinder is only a round Prism, being constituted of an infinite Series of equal Circles; that of its Base or End being one of the Terms, and its Height *DB* is the Number of all the Terms. Therefore the Area of its Base *BA* being multiply'd into *DB*, will be its Solidity, by *Lemma* 1.

Viz. Let $D = BA$ and $H = GA$.

Then $0{,}7854\, DD \times H =$ its Solidity.

Example,

Example, Let the Diameter of its Base be $D = 16$, and its Height $H = 42$.

Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the Area of its Base.

And $201,0624 \times 42 = 8444,6208$ the solid Content of that Cylinder *DBGA*.

Corollary.

Hence it is evident, that every Square Parallelopipedon is to its inscrib'd Cylinder, As 1 : is to 0,7854. Or in whole Numbers, as 452 : to 355 very near.

And that all Prisms are in Proportion to their inscrib'd Cylinders, as the Area's of their Bases are.

THEOREM XII.

The Curve Superficies of every Right Cylinder, is equal to the Rectangle made of its Height into the Periphery of its Base.

That is, *DB* multiply'd into the Periphery of the Diameter *BA*, will produce the Curve Superficies of the last Cylinder *DGBA*.

For the Cylinder is constituted of an infinite Series of equal Circles; (according to the last *Theor.*) therefore its Curve Superficies is compos'd of the Peripheries of those Circles, by Definition 20. But the Periphery of its Base *BA* is one of the Terms, and its Height *DB* is the Number of Terms. Therefore, &c. As by *Lemma* 1.

To which, if there be added the Area's of both its Ends, (or Bases) the Sum will be the Superficies of the whole Cylinder.

Example, Suppose the Diameter of its Base to be $BA = 16$, and its Height $DB = 42$. As before.

Then $1 : 3,1416 :: 16 : 50,2656$ the Periphery of its Base.

Again, $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the Area of each End or Base.

Then $50,2656 \times 42 = 2111,1552$ the Curve Superficies.
To which add $201,0264 \times 2 = 402,1248$ the Area's of both Ends.

The Sum $= 2513,2800$ is the Superficies of the whole Cylinder.

THEOREM XIII.

Every Cone is the third Part of a Cylinder, having the same Base with it, and their Altitudes equal. (10. *e.* 12.)

Demon-

Demonstration.

The Truth of this Theorem may be easily conceiv'd by only considering, that a Cone is but a round Pyramid, and therefore it must needs have the same Ratio to its circumscribing Cylinder, as the Square Pyramid hath to its circumscribing Parallelopipedon; *viz.* as 1 to 3. However, to make it yet clearer, let it be farther consider'd, That

Every Right Cone is constituted of an infinite Series of Circles, whose Diameters do continually encrease in Arithmetical Progression, beginning at the Vertex or Point *V*, the Area of its Base *BA* being the greatest Term, and its Perpendicular Height *VC* the Number of all the Terms; therefore the Area of the Circle *BA* $\times \frac{1}{3}$ *VC* will be the Sum of all the Series, by *Lemma* 3. which is the Cone's Solidity.

Example. Let the Diameter of its Base be $BA = 16$, and its Height $VC = 42$.

Then $1 : 0{,}7854 :: 16 \times 16 = 256 : 201{,}0624$ the Area of the Base.

And $\frac{201.0624 \times 42}{3} = 2814{,}8736$, the Solidity of the Cone *BVA*.

Or thus, $201{,}0624 \times \frac{42}{3} = 2814{,}8736$, &c.

Corollary.

Hence it follows, that every Square Pyramid is to its inscrib'd Cone, as 1 : 0,7854. (Or, as 452 : 355.) Consequently, that all Pyramids have the same Ratio to their inscrib'd Cones, as the Areas of their Bases have.

THEOREM IV.

The Curve Superficies of every Right Cone, is equal to half the Rectangle of the Periphery of its Base into the Length of its Side.

The Truth of this Theorem is self-evident from the Definition of a Cone *Chap.* 1. *Part* 4. where it appears, that the Curve Superficies of every Right Cone (as *BVA*) is equal to the Area of a Sector of that Circle, whose Radius is the Side of the Cone (*VB*) and its Arch equal to the Periphery of the Cone's Base (*B.A.*) But the Area of any Sector is equal to half the Rectangle of the Radius into its Arch, by Theorem 4. Therefore, *&c.*

Example, Suppose the Length of the Cone's Side to be *VB*, or *VA* = 42,7551.
And the Diameter of its Base, *BA* = 16. As before.
Then will 50,2656 be the Periphery of its Base.

And $\frac{50,2656 \times 42,7551}{2}$ = 1074,5553 &c. the Curve of the Superficies.

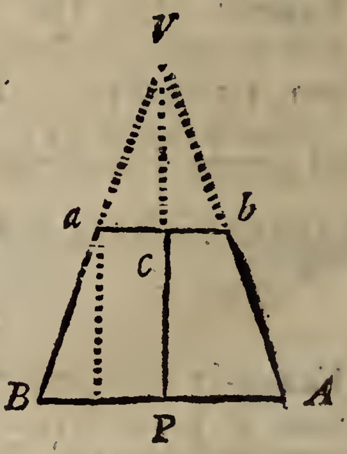
To which, if there be added the Area of its Base, the Sum will be the Superficies of the whole *viz.* all the) Cone.

That is 1074,5553
+201,0624 the Area of the Base.
Sum 1275,6177 is the total Superficies. &c.

Note, *The Truth of this Theorem may be prov'd from the Consideration of the last Theorem, and Definition* 20.

SCHOLIUM.

From the 10th and 13th Theorems may be easily deduced several Theorems for finding the solid Content of any Frustum or Part, either of a Pyramid or Cone, by a plane Parallel to its Base.



Suppose a square Pyramid, as *BVA*, to be cut by a Plane at *a b*, parallel to its Base *BA* and it were requir'd to find the Solidity of the Frustum or Part *a b A B*. Let there be given.

D=*BA* the Side of the greater Base.
d=*ba* the Side of the lesser Base.
H=*CP* the Perpendicular Height,

First	1	$D-d : H :: d : \frac{dH}{D-d} = VC$ by the Figure.
Then	2	$DD \times \frac{H + VC}{3}$ = the whole Pyramid *BVA*. By Theorem 10.
And	3	$dd \times \frac{1}{3} VC$ = the Pyramid *aVb* cut off.
Viz. 1, 2	4	$\frac{DDDH}{3D - 3d}$ = the whole Pyramid *BVA*.
And 1, 3	5	$\frac{dddH}{3D - 3d}$ = the Pyramid *aVb*.
4 — 5	6	$\frac{DDDH - dddH}{3D - 3d}$ = the Frustum *abAB*.
6, *Reduc.*	7	$DD + Dd + dd : \times \frac{1}{3} H$ = the Frustum *abAB*.

Which in Words gives this following Theorem.

THEOREM

THEOREM XV.

To the Rectangle of the Sides of the two Bases, add the Sum of their Squares; that Sum being multiply'd into one Third of the Frustum's Height, will give its Solidity.

Example, Suppose the Side of the greater Base $BA = 16$.
And the Side of the lesser Base (or Top) $ab = 12$.
The Height $CP = 9$.
Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$.
Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$.
Or $592 \times \frac{9}{3} = 1776$, the Content of the Frustum of a square Pyramid.

And if it were the like Frustum of a right Cone, it may be found by the same Theorem. Supposing D equal the Diameter of the greater Base, $d =$ the Diameter of the lesser, and $H =$ the Height of the Frustum.

Then seeing the Sum of all the Squares which constitute the Frustum of a square Pyramid, are to the Sum of all the Circles which constitute the like Frustum of a right Cone, in the Ratio of 1 : to 0.7854. (or of 452 : to 355.) Therefore,
it will be $1 : 0.7854 :: DD + Dd + dd \times \frac{1}{3} H : 0,7854\, DD + 0,7854\, Dd + 0,7854\, dd \times \frac{1}{3} H =$ the Cone's Frustum.

That is, in the last Example, $1 : 0,7854 :: 1776 : 1394,8704$ the like Frustum of a right Cone.
Or, because $\frac{1}{0,7854} = 1.273236$, &c. Therefore it may be made $1,273236)\ DD + Dd + dd \times \frac{1}{3} H$ (= the same Frustum.
That is, $1,273236)\ 1776\ (1394,87$. &c. As before.

And if you take the Triple of this Divisor, *viz.* $1,273236 \times 3$ it will be $3,8197)\ DD + Dd + dd : \times H$ (= the Frustum, &c.

Again.

Suppose	1	$x = D - d$. And $F =$ the Frustum.
Then	2	$DD + Dd + dd = \frac{3F}{H}$, by 7th Step of the last.
1 ⊖ 2	3	$xx = DD - 2Dd + dd$
2 — 3	4	$3Dd = \frac{3F}{H} - xx$
4 ÷ 3	5	$Dd = \frac{F}{H} - \frac{1}{3} xx$. Or $DD + \frac{1}{3} xx = \frac{F}{H}$
5 × H	6	$Dd + \frac{1}{3} xx : \times H = F$ the Frustum $abAB$.

Hence we have another easy Theorem for finding the same Frustum.

THEOREM XVI.

To the Rectangle of the Sides of the two Bases, add one Third Part of the Square of their Difference; that Sum being multiply'd into the Height, will produce the Solidity.

Example, Let $D = 16$. $d = 12$. and $H = 9$. As before.

Then $Dd = 192$. $D - d = 4 = x$. $\frac{1}{3}xx = \frac{4 \times 4}{3} = 5,3333$.

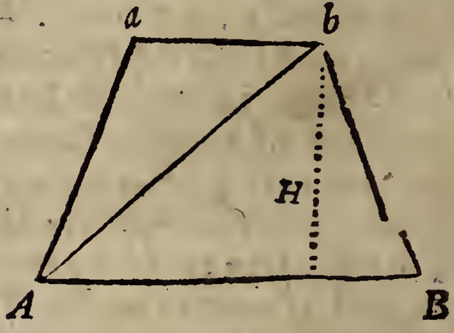
And $192 + 5,3333 = 197.3333$.

Lastly, $197,3333 \times 9 = 1775,9997$ the Solidity of the Frustum of the square Pyramid. As before.

And 3,81968) 1775,9997 (1394,87, &c. the like Frustum of a right Cone. As before.

Either of the two last Theorems (being rightly apply'd) will produce the true solid Content of all Frustums of any kind of Pyramids that are intercepted between two parallel and alike Planes or Bases. As above.

But if such Frustums are cut thro' the Extremities of both Bases by a Diagonal Plane (as Ab in the annex'd Figure) into two Parts, Aab, and ABb, call'd Hoofs; then the Solidity of those Hoofs is usually found by dividing the middle Term Dd of the Equation $DD + Dd + dd$ into two Parts, and adding one of those Parts to the Square of each Base.



Thus, $DD + \frac{1}{2}Dd : \times \frac{1}{3}H =$ the great Hoof ABb.
And $dd + \frac{1}{2}Dd : \times \frac{1}{3}H =$ the lesser Hoof Aab of the Frustum of any square Pyramid.

Then 3,8197) $DD + \frac{1}{2} Dd : \times H$ (= the greater Hoof of a Cone.
And 3,8197) $dd + \frac{1}{2} Dd : \times H$ (= the lesser Hoof, &c.

These are the Theorems made use of by Mr. *Dary* in his Book of *Gaging*, and are pretty near the Truth, but not exactly so; for they give the Solidity of the upper Hoof Aab a small matter too big, and the lower Hoof ABb as much too little.

Now in order to rectify that small Error, I shall here propose the two following Theorems, which come very near the Truth, and are more easily perform'd than those proposed in the first Impression of this Book.

First,

First, $DD + \frac{1}{2}Dd + D - d : \times \frac{1}{3}H$ will be the Solidity of the greater Hoof *ABb*.

Secondly, $dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H$ will give the Solidity of the lesser Hoof *Aab*, of the Frustum of any square Pyramid.

And for the like Hoofs of the Frustum of any right Cone, it will be

Thus, $3{,}8197) DD + \frac{1}{2}Dd - D - d : \times \frac{1}{3}H (=$ the greater Hoof.
And $3{,}8197) dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H =$ the lesser Hoof.

Note, *In order to avoid many Words in the following Demonstrations, let* ⊙ *signify any Circle in general; and if any two Letters be joyn'd to it, thus,* ⊙ *B A* &c. *it then denotes the Area of such a Circle as those two Letters represent the Radius of.*

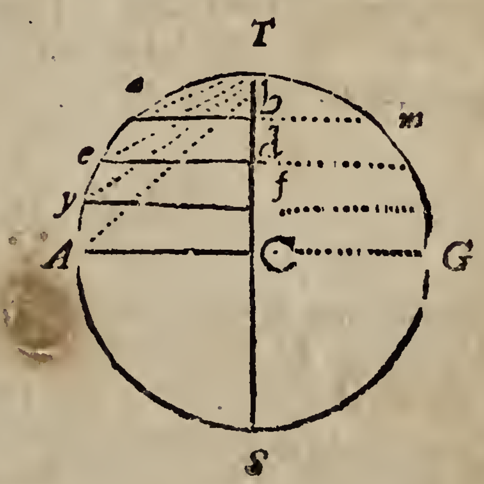
THEOREM XVII.

The Superficies of every Sphere (or Globe) is equal to four times the Area of its greatest Circle.

That is, of a Circle whose Diameter is the Axis of the Sphere.

Demonstration.

If any Semi-circle (as *ATGS*) be turn'd or moved about its Diameter (*TS*) it will describe a solid Body call'd a Sphere, which will be constituted of an infinite Series of concentrick or parallel Circles, whose Diameters are Chords, *viz.* ⊙*ab*, ⊙*ed*, ⊙*ef*, &c. by *Definition* 14.



Consequently, the Superficies of the Sphere will be composed of the Peripheries of those Circles which constitute its Solidity. By *Definition* 20.

Let $D = TS$, the Axis of any Sphere. Then, according to the Property of a Circle, it

will be	1	$D - Tb \times Tb = \square\, ab$
That is	2	$D \times Tb - \square\, Tb = \square\, ab$
Therefore	3	$D \times Tb = \square\, aT$. For $\square\, ab + \square\, Tb = \square\, aT$
And	4	$D \times Td = \square\, eT$
	5	$D \times Tf = \square\, yT$, &c.

Hence

Hence it is evident, that the Series □ *aT*. □ *eT*. □ *yT* &c. are in the same *Ratio* with *Tb*, *Td*, *Tf*, &c. *viz*. in Arithmetical Progression. Whence it follows, that the ⊙ *aT* = the Sum of all the Circle's Peripheries between *T* and *b*,

And the ⊙ *eT* = the Sum of all the Circle's Peripheries between *T* and *d*, &c.

Consequently, that the ⊙ *AT* = the Sum of all the Circle's Peripheries included between *T* and *C* That is, ⊙ *AT* = the Superficies of the Semi-sphere.

And because the □ *AC* + □ *TC* = □ *AT*, and □ *AC* = □ *TC*. Therefore ⊙ *AT* = 2 ⊙ *AC* is the Superficies of the Semi-sphere.

Consequently, 4 ⊙ *AC* will be the Superficies of the whole Sphere, Q. E. D.

Example. Suppose the Axis *TS* = *D* = 16. Then *DD* = 256. And 1 : 0,7854 :: 256 : 201,0624 = ⊙ *AC*. For $\frac{1}{2}$ *D* = *AC* Then 201,0624 × 4 = 804,2496 the Superficies of the whole Sphere.

Or, because 3 1416 is four times 0 7854, therefore it will always be 1 : 3,1416 :: *DD* : 3,1416 *DD* the Superficies of the Sphere, (as before) and it's equal to the Curve Superficies of a right Cylinder, whose Diameter and Height are each = the Axis of the Sphere.

For 3,1416 *D* = the Periphery of the Cylinder's Base, and that multiply'd with *D* its Height, will be 3,1416 *DD* the Curve Superficies of the Cylinder, by *Theor*. 12.

And if to this there be added the Areas of its two Bases, (or Ends) *viz*. 1,5708 *DD*. Then it is evident, that the whole Superficies of the Cylinder will be to that of the Sphere, in the Proportion of 3 to 2.

SCHOLIUM.

From the Method here used in proving the last *Theorem*, it will be easy to find the Curve Superficies of any Segment or Part of a Sphere, that is cut off by a right Line, or Plane; *viz*. such as the Segment *aTm* in the last Scheme, whose Curve Superficies is ⊙ *aT*, (as above.) Therefore (because □ *ab* + □ *Tb* = □ *aT*) it will be ⊙ *ab* + ⊙ *Tb* = the Curve Superficies of that Segment.

But if the Axis *TS*, and Height *Tb* of the Segment are given, then it will be *TS* × *Tb* = □ *aT* as in the third Step above. Which gives this Proportion or Theorem.

Viz.

Viz. { *As the Axis of the Sphere : is to the whole Superficies of the Sphere :: so is the Height of any Segment : to its Curve Superficies*

To which, if there be added the Area of the Segment's Base, the Sum will be the Superficies of the whole Segment.

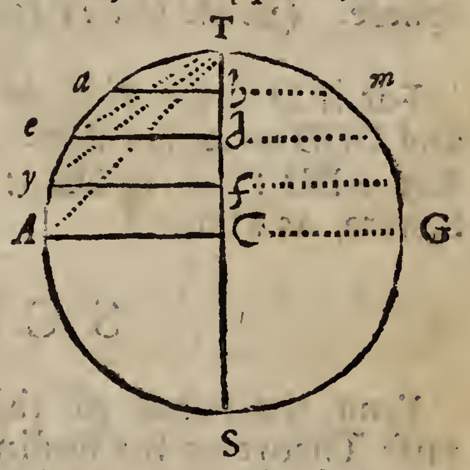
THEOREM XVIII.

Every Sphere is equal to two Thirds of its Circumscribing Cylinder.

That is, of a Cylinder, whose Height and Diameter of its Base are each equal to the Axis of the Sphere.

Demonstration.

According to the Work in the last Theorem, it appears, that $\odot\, ab.\ \odot\, ed.\ \odot\, yf$,&c. do constitute the Solidity of the Sphere, and that $\Box\, aT$, $\Box\, eT$, $\Box\, yT$, &c. are a Series of Terms in *Arithmetical Progression*, $\Box\, AT$ being the greatest Term, and TC the Number of Terms. Therefore $\odot\, AT \times \frac{1}{2} TC =$ the Sum of all the Series. *Per Lemma* 2.



And because $\Box\, aT - \Box\, Tb = \Box\, ab$. $\Box\, eT - \Box\, Td = \Box\, ed$. $\Box\, yT - \Box\, Tf = \Box\, yd$. $\Box\, AT - \Box\, TC = \Box\, AC$, &c. wherein $\Box Tb$, $\Box Td$, $\Box Tf$ &c. are Series of Squares. whose Roots Tb, Td, Tf, are in *Arithmetical Progression*; $\Box\, TC$ being the greatest Term, and TC the Number of Terms. Therefore $\odot TC \times \frac{1}{3} TC =$ the Sum of all that Series. *Per Lemma* 3.

Consequently, $\odot AT \times \frac{1}{2} TC : - \odot\, TC \times \frac{1}{3} TC =$ the Sum of the Series $\odot\, ab.\ \odot\, ed.\ \odot\, yf.$ &c. which constitute the Solidity of the half Sphere ATG. Put $D = 2TC$ the Axis of the Sphere. Then $\frac{1}{4} D = \frac{1}{2} TC$, and $\frac{1}{6} D = \frac{1}{3} TC$. And because $\Box\, AT = 2\, \Box\, TC$; therefore $\odot AT = 2 \odot TC = 1{,}5708\, DD$. And $1{,}5708\, DD \times \frac{1}{4} D = 0{,}3927\, DDD$.

Again, $\odot TC \times \frac{1}{3} TC = 0{,}7854\, DD \times \frac{1}{6} D = 0{,}1309\, DDD$.

Then $0{,}3927\, DDD - 0{,}1309\, DDD = 0{,}2618\, DDD$ the Solidity of the Semi-sphere ATG.

Consequently. $0{,}2618\, DDD \times 2 = 0{,}5236\, DDD$ will be the solid Content of the whole Sphere which is equal to $\frac{2}{3}$ of the Cylinder, whose Diameter of its Base and Height $= D$.

For $0{,}7854\, DDD =$ the Solidity of the Cylinder, by *Theorem* II. But $\frac{2}{3}$ of $0{,}7854\, DDD = 0{,}5236\, DDD$, as before.

Therefore, *&c.* as by *Theor.*

Example

Example, Suppose the Axis D = 16. then DDD = 4096, and 1 : 0,5236 :: 4096 : 2144,6656 the solid Content of that Sphere.

Corollaries.

1. Hence it appears, that the solid Content of every Sphere is equal to its Superficies multiply'd into one sixth Part of its Axis.

For its Superficies is 3,1416 DD, by *Theorem* 17.
But $3,1416\ DD \times \frac{1}{6} D = 0,5236\ DDD$ the solid Content, as before.

2. And hence it is also evident, that there is the like *Ratio* or Habitude between the Cube and its inscribed Sphere, as is between the Square and its inscrib'd Circle. And that is, As the Superficies of any Cube : is to the Superficies of its inscrib'd Sphere :: so is the solid Content of that Cube : to the solid Content of the Sphere. (*See the Circle's Proportion, Page* 407.)

For if D = the Side of the Cube, then 6 DD = its Superficies, and DDD = its Solidity. And 3,1416 DD = the Sphere's Superficies. But 6 DD : 3,1416 DD :: DDD : 0,5236 DDD the Solidity of the Sphere, as above.

SCHOLIUM.

From the Proof of this Theorem, it will be easy to deduce or raise Theorems for finding the solid Content of any Frustum or Segment of a Sphere; as *aTm* in the last Figure.

For we there suppose the Segment *aTm* to be constituted of an infinite Series of Circles, which have the same *Ratio* with all those Circles that constitute the *Semi-sphere*.

Therefore it follows, that $\odot\, aT \times \frac{1}{2} Tb : - \odot\, bT \times \frac{1}{3} Tb$ will be will be the Sum of all the Circles intercepted between T and *b*. Consequently it will be Solidity of that Segment.

And because the $\square\, ab + \square\, Tb = \square\, aT$: Therefore,

$\odot\, ab + \odot\, Tb \times \frac{1}{2} Tb - \odot\, Tb \times \frac{1}{3} bT =$ the same Solidity.

Let $c = ab$, half the Segment's Base; $h = Tb$, its Height; and $S =$ the Solidity of the Segment or Frustum.

Then $\odot\, ab = 3,1416\, cc$. and $\odot$ TB $= 3,1416\, hh$.

Consequently, $\dfrac{3,1416\, cch + 3,1416\, hhh}{2} - \dfrac{3,1416\, hhh}{3} = S$

which being reduced will become $3cch + hhh \times 0,5236 = S$.

Or 1,909855) $3cch + hhh$ ($= S$. For 0,5236) 1 0000 (1,909855 which is one Theorem for finding the Frustum's Solidity.

Note,

Note, Here we suppose the Height of the Segment, and the Diameter of its Base to be given: But if the Axis of the Sphere, and the Height of the Segment be given, then putting $D =$ the Sphere's Axis, $b =$ the Segment's Height, and c as before, it will be $D - b \times b = cc$. Viz. $Db - bb = cc$.

Therefore $3Dbb - 2bbb = 3ccb + bbb$.

Consequently, $3Dbb - 2bbb \times 0{,}5236 = S$ the Frustum's Solidity.

Or $1{,}90985)\ 3Dbb - 2bbb\ (= S$. As before.

Which is a second Theorem for finding the same Frustum aTm.

And if it be requir'd to find the middle Part $amNK$, usually call'd the Middle Zone of a Sphere; then, because it is supposed that $am = NK$, or which is all one, that $bC = CB$; therefore it is plain, that if twice the Segment aTm be taken from the Solidity of the whole Sphere, there will remain the Middle Zone $amNK$.

But because that Work is a little troublesom, I shall here shew how to raise a Theorem for doing it.

First, Because $AC = yC = eC = aC = TC$. Therefore it will be $\Box AC - \Box Cf = \Box yf$. $\Box AC - \Box Cd = \Box ed$. $\Box AC - \Box Cb = \Box ab$, &c.

Here because $\Box AC$. $\Box AC$. $\Box AC$, &c. are a Series of Equals, and Cb the Number of all the Terms; therefore $\Box AC \times Cb =$ the Sum of all the Series, by *Lemma* 1.

And $\Box Cf$. $\Box Cd$. $\Box Cb$. &c. being a Series of Squares, whose Roots are in *Arithmetical Progression*, beginning at the Center or Point C, viz. 0, Cf, Cd, Cb, wherein the greatest Term is $\Box Cb$ and, Number of Terms is Cb. *Ergo* $\Box Cb \times \frac{1}{3} Cb =$ the Sum of all the Series, by *Lemma* 3.

Consequently, the $\odot AC \times Cb : - \odot Cb \times \frac{1}{3} Cb =$ the Sum of all the Series $\odot yf$. $\odot ed$. $\odot ab$, which do constitute the Solidity of the half Zone $amAG$.

And because $\Box AC - \Box Cb = \Box ab$. *Ergo* $\odot AC - \odot ab = \odot Cb$.

Consequently, $\odot AC \times Cb : - \dfrac{\odot AC + \odot ab : \times CB}{3} = 2\odot AC + \odot ab : \times \frac{1}{3} Cb$ will be the Solidity of the half Zone.

Put $D = AG = 2AC$. $x = am$. and $H = bB = 2Cb$.

Then $\odot AC = 0{,}7854\,DD$. $\odot ab = 0{,}7854\,xx$. And if we turn the common Factor $0{,}7854$ into the Divisor $1{,}27323$, and then take

Triple of that Divisor, viz. 3,8197 (as before in the Frustums of Pyramids) the Result of the preceding Work will produce this following Theorem.

Theorem IX. $\left\{\frac{2DD + xx}{3,8197} : \times H = \right\}$ the middle Zone *amNK*.

THEOREM XX.

Spheres are in Proportion one to another, as the Cubes of their Diameters. (18. *e.* 12.)

Demonstration.

Suppose D = the Diameter or Axis of any Sphere, and *d* = the Dianeter of another Sphere, either greater or lesser.

Then is 0,5236 DDD = the Solidity of one Sphere, and 0,5236 *ddd* = the Solidity of the other Sphere, by *Theorem* 18.

But DDD : *ddd* : : 0,5236 DDD : 0,5236 *ddd*. Q. E. D.

THEOREM XXI.

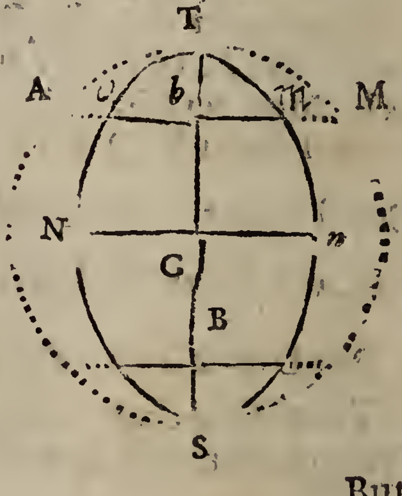
The solid Content of every Spheroid is equal to two Thirds of its circumscribing Cylinder.

Demonstration.

Suppose the Figure NT*n*SN in the annex'd Scheme to represent a Spheroid form'd by the Rotation of the Semi-Ellipsis TNS about its transverse Axis TS, (as by Definition 15.)

Let D = TS the Length of the Spheroid, and the Axis of its circumscribing Sphere. And *d* = N*n* the Diameter of the greatest Circle of the Spheroid.

Then because □TC : □NC : : □A*b* : □*ab*, by Step 3. in Theor. 7.
Therefore it will be DD : *dd* : : □A*b* : □*ab* : : ⊙A*b* : ⊙ *ab* &c.



But the Sum of an infinite Series of such Circles as ⊙ A *b*, (whose Diameters are Chords) do constitute the Solidity of the Sphere, (as before at Theorem 18.) And the Sum of an infinite Series of such Circles, as ⊙ *a b*, (*viz.* whose Diameters are Ordinates of the Ellipsis) do constitute the Solidity of the Spheroid, by Definition 15.

Ergo DD : *dd* : : 0,5236 DDD : 0,5236 *dd* D = the Solidity of the Spheroid, by *Lem.* 6.

But

But $,05236\,ddD = \frac{2}{3}$ of the Cylinder, whose Diameter is $= d$, and Height $= D$, by Theorem 11. Q. E. D.

Now from this Proportion between the Sphere and its inscrib'd Spheroid, it will be very easy to deduce Theorems for finding the solid Content, either of the Segment or middle Zone of any Spheroid, having the same Height with that of the Sphere.

For $\left\{\right.$ *As the Solidity of the whole Sphere : is to the Solidity of the whole Spheroid : so is any Part of the Sphere : to the like Part of the Spheroid, by* Conv. to Lem. 6.

As for Instance, Suppose it were requir'd to find the middle Zone of any Spheroid.

Let $D = TS$, and $d = Nn$, as above; and $H = bB$. $x = AM$, as in Theorem 19. And let $c = am$.

Then $\left\{ \dfrac{2DD + xx}{3,8197} \times H \right.$ = the middle Zone of the Sphere. And

$$0,5236\,DDD : 0,5236\,ddD :: \frac{2DD + xx}{3,8197} \times H : \frac{2dd \times H}{3,8197} + \frac{xxdd \times H}{3,8197\,DD} =$$

the middle Zone of the Spheroid.

Again, $DD : dd :: xx : cc$. Therefore $\dfrac{xxdd}{DD} = cc$.

Consequently, $\dfrac{xxdd}{DD} \times \dfrac{H}{3,8197} = \dfrac{cc}{3,8197} \times H$. Which being taken instead of $\dfrac{xxdd \times H}{3,8197\,DD}$ there will arise this following

Theorem XXII. $\left\{ \dfrac{2dd + cc}{3,8197} \right. \times H = \left\{ \right.$ the middle Zone of the Spheroid, being the very same with Theorem 19.

Note *In the same manner you may raise Theorems for finding the Segment of a Spheroid, cut off at either of its Ends, &c.*

THEOREM XXIII.

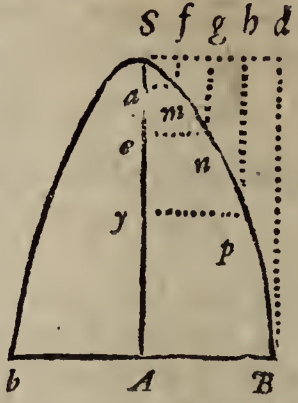
The Area of every Parabola is equal to two Thirds of its circumscribing Parallelogram.

Demonstration.

Let the Figure *SAB* represent half a Parabola; make *DB* parallel to the Axis *SA*, and *SI* parallel to the Semi-Ordinate *AB*. And suppose *SI* to be divided into an infinite Series of equidistant

Points, as f, g, h &c. and from those Points imagine a Series of parallel Lines, *viz.* fm, gn, hp, &c. to touch the Curve of the Parabola, and meet the Semi-Ordinates ma, ne, yp, &c.

Then according to the Property of the Parabola it will



be { 1 | $SA : \Box AB :: Sa : \Box am$.
2 | $SA : \Box AB :: Se : \Box en$
3 | $SA : \Box AB :: Sy : \Box yp$, &c. }

But $Sa = fm$. $Se = gn$. $Sy = hp$. $SA = dB$

Therefore alternately it will be

3, | 4 | $\Box AB : dB :: \Box yp : hp$
2, | 5 | $\Box AB : dB :: \Box en : gn$
1, | 6 | $\Box AB : dB :: \Box am : fm$

In these Proportions $\Box am$, $\Box en$, $\Box yp$, &c. are a Series of Squares, whose Roots, Sf, Sg, Sh, &c. are in *Arithmetical Progression*, beginning at the Point S. And because the Lines hp, gn, fm, &c. have the same *Ratio*, therefore they are as such a Series of Squares, wherein dB is the greatest Term, and Sd the Number of Terms.

Consequently, $\dfrac{dB \times Sb}{3} =$ the Sum of all those Lines, by *Lemma* 3.

But $SA \times AB = dB \times Sd$. Therefore $\dfrac{SA \times AB}{3} =$ the Sum of all that Series of Lines; but all those Lines do constitute the Area of the Semi-Parabola's Complement, *viz.* the Area of what half the Parabola SAB wants of completing or filling up the Prarallelogram $Sd\ AB$.

Wherefore $SA \times AB : - \frac{1}{3} SA \times AB = \dfrac{2\,SA \times AB}{3}$ will be the Area of half the Parabola SAB.

Consequently, $\frac{2}{3} SA \times bB$, will be the Area of the whole Parabola bSB.

Example. Suppose the Base or greatest Ordinate of a Parabola to be $bB = 24$ and its intercepted Diameter or Axis be $SA = 33$. Then $2\,SA \times bB = 66 \times 24 = 1584$. and 3) 1584 (528 the Area of that Parabola.

THEOREM XXIV.

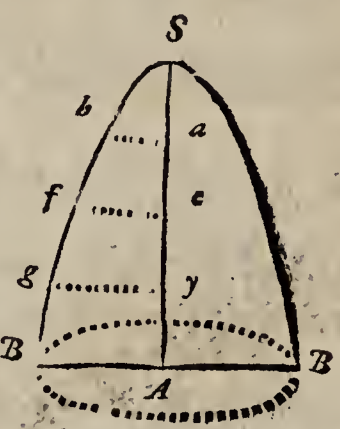
Every Parabolic Conoid is equal to one half of its circumscribing Cylinder.

Demon-

Demonstration.

If any Semi-parabola (as *BSA*) be turn'd or mov'd about its Axis, (*SA*) it will form a solid Parabolic Conoid, constituted of an infinite Series of Circles, viz. ⊙ *ba*, ⊙ *fe*, ⊙ *gy*, &c. by Defin. 17.

Now, according to the Property of every Parabola, it will be, $SA : AB :: AB : \frac{\square AB}{SA} = L$, the *Latus Rectum*.



$$\text{Then} \begin{cases} Sa \times L = \square\, ba \\ Se \times L = \square\, fe \\ Sy \times L = \square\, gy, \text{ \&c.} \end{cases}$$

Here $Sa \times L$, $Se \times L$, $Sy \times L$, &c. are a Series of Terms in *Arithmetical Progression*. Therefore $\square\, ba$, $\square\, fe$, $\square\, gy$, &c. are also a Series of Terms in the same Progression, beginning at the Point *S*, wherein $\square AB$ is the greatest Term, and *SA* the Number of all the Terms. Therefore $\square AB \times \frac{1}{2} SA =$ the Sum of all the Series, by *Lemma* 2.

Consequently, $\odot AB \times \frac{1}{2} SA =$ the Sum of all the Series of ⊙ *ba*, ⊙ *fe*, ⊙ *gy*, &c. which do constitute the Solidity of the Conoid.

And putting $D = 2AB$ and $H = SA$.

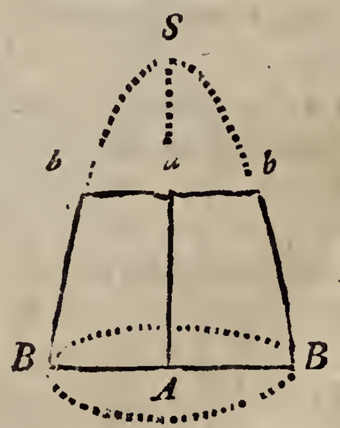
Then, $0{,}7854\, DD \times \frac{1}{2} H = 0{,}3927\, DDH$ will be the solid Content of the Conoid; which is just half the Cylinder, whose Base $= D$, and Height $= H$. See *Theorem* 11. Q. E. D.

This being understood, it will be easy to raise a Theorem for finding the Lower Frustum of any Parabolic Conoid.

For supposing $b = aA$ the Height of the Frustum, and $p = Sa$ the Height of the Part *bSb* cut off. Then $b + p = SA$ the Height of the whole Conoid.

Consequently, $\frac{\odot AB \times b + \odot AB \times p}{2} =$ the Solidity of the whole Conoid.

And $\frac{\odot ba \times p}{2} =$ the Solidity of the Part cut off.



Ergo	1	$\frac{\odot AB \times b + \odot AB \times p - \odot ba \times p}{2} =$ the Solidity of the Frustum.
But	2	$b + p : \square AB :: p : \square\, ba$
Conseq.	3	$b + p : \odot AB :: p : \odot\, ba$
3 ∴	4	$\odot AB \times p = \odot ba \times b + \odot ba \times p$

4—⊙

$4 - \odot ba \times p$	5	$\odot AB \times p :- \odot ba \times p = \odot ba \times b$
1×2	6	$\odot AB \times b : + \odot AB \times p :- \odot ba \times p = 2F$
$6 - 5$	7	$\odot AB \times b = 2F :- \odot ba \times b$
$7 + \odot ba \times b$	8	$\odot AB + b : + \odot ba \times b = 2F$
$8 \div 2$	9	$\frac{\odot AB + \odot ba}{2} : \times b = F$ the Fruſtum's Solidity

Let $D = AB$ as before, and $d = 2\,ba$ the Diameter of the Part cut off. Then we ſhall have this following

Theorem XXV. $\left\{ 0{,}3927\,DD + 0{,}3927\,dd : \times b = \right.$ the Solidity of the Fruſtum requir'd.

Or $\left\{ \frac{DD + dd}{2{,}5464} \right. \times b$ the Fruſtum. For ,3927) 1,0000 (= 2,5464

And becauſe $2{,}5464 + \frac{2{,}5464}{2} = 3{,}8196$. Therefore it may be made $3{,}8196)\,DD + dd : \times \frac{1}{2}b$ (= the ſame Fruſtum, &c.

Note, *The Reaſon why I have reduced this Theorem to have the ſame Diviſor with thoſe of the Fruſtums of Pyramids,* &c. *will beſt appear farther on,* viz. *when they all come to be apply'd to Practice in Gaging.*

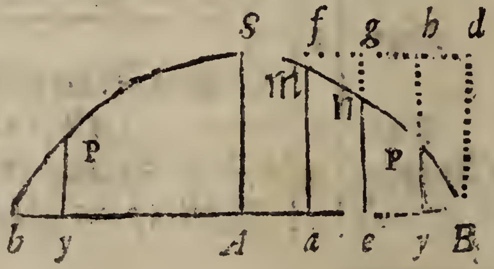
THEOREM XXVI.

Every Parabolic Spindle (*or* Pyramidoid) *is equal to eight Fifteenths of its circumſcribing Cylinder.*

Demonſtration.

If any Acute Parabola, as *bSB* be turn'd or mov'd about its greateſt Ordinate *bAB* it will form a Solid, call'd a Parabolic Spindle, conſtituted of an infinite Series of ⊙ *ma*, ⊙ *ne*, ⊙ *py*, &c. by Definition 18.

Let us ſuppoſe the Line *Sd* parallel to *AB*, &c. As at *Theorem* 23. Then it hath already been prov'd, that the Lines *fm*, *gn*, *hp*, &c. are a Series of Squares, whoſe Roots are in *Arithmetical Progreſſion*. Conſequently, their Squares, *viz.* □ *fm* □ *gn*, □ *hp*, &c. will be a Series of Biquadrates, whoſe Roots will be in *Arithmetical Progreſſion*; which being premiſed, we may proceed thus.



First	1	$SA - fm = ma$
	2	$SA - gn = ne$
	3	$SA - hp = py$, &c.

1 ⊙ 2

1 ⊕ 2	4	$\square SA - 2SA \times fm + \square fm = \square ma$
2 ⊕ 2	5	$\square SA - 2SA \times gn + \square gn = \square ne$
3 ⊕ 2	6	$\square SA - 2SA \times hp + \square hp = \square py$, &c.

1. In these Æquations, the $\square SA$, $\square SA$, $\square SA$ being a Series of Equals, and AB the Number of all the Terms; therefore it will be $\square SA \times AB =$ the Sum of all the Series, by *Lemma* 1.

2. Because fm, gn, hp, &c. are as a Series of Squares, wherein SA is the greatest Term, and AB the Number of all the Terms. Therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$ will be the Sum of all that Series, by *Lemma* 3.

3. And the $\square fm \cdot \square gn \cdot \square hp$, &c. will be a Series of Terms in the *Ratio* of Biquadrates, as above, $\square dB = \square SA$ being the greatest Term, and AB the Number of all the Terms; therefore it will be $\frac{\square SA \times AB}{5} =$ the Sum of all the Series, by *Lemma* 5.

Wherefore it follows, that $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5}$ $=$ the Sum of all the Series of $\square ma$, $\square ne$, $\square py$, &c.

That is, $\frac{8\square SA \times AB}{15} =$ the Sum of all the Series of $\square ma \cdot \square ne \cdot \square hp \cdot \square dB$, &c.

Consequently, $\frac{8 \odot SA \times AB}{15} =$ the Sum of all the Series of $\odot ma \cdot \odot ne \cdot \odot py \cdot$ &c, which do constitute the Solidity of half the Spindle, *viz.* of SAB.

Therefore, putting $D = 2SA$, and $H = 2AB$ (viz. bAB) it will be $0{,}41888\,DDH =$ the Solidity of the whole Parabolic Spindle bSB, being $\frac{8}{15}$ of $0{,}7854\,DDH$ the Solidity of its circumscribing Cylinder. Q. E. D.

From hence we may also raise a *Theorem* for finding the Frustum $SApy$ of the last Figure.

For $\odot SA$ being the greatest Term, $\odot py$ the least Term, and Ay the Number of all the Terms and Circles included between A and y.

Therefore	1	$\left\{ \square SA - \frac{2SA \times hp}{3} + \frac{\square hp}{5} : \times Ay = z \right.$ the Sum of all the Series $\square SA$, $\square ma$, $\square gn$, $\square py$.
1 × 3	2	$3\square SA - 2SA \times hp + \frac{3\square hp}{5} : \times Ay = 3z$

2 ÷

$2 \div Ay$	3	$3\square SA - 2SA \times hp + \frac{3\square hp}{5} = \frac{3z}{Ay}$
But	4	$\square SA - 2SA \times hp = \square py - \square hp$. By 6th Step.
$3 - 4$	5	$2\square SA + \frac{3\square hp}{5} = \frac{3z}{Ay} - \square py + \square hp$
$5 \pm$ &c.	6	$2\square SA + \square py - \frac{2}{5}\square hp = \frac{3z}{Ay}$
Conseq.	7	$2\odot SA + \odot py - \frac{2}{5}\odot hp : \times \frac{1}{3} Ay = z$. the Sum of all

the Series of $\odot SA$. $\odot ma$. $\odot ne$. $\odot py$, which do constitute the Solidity of the Frustum $SApy$. Therefore putting $D = 2SA$ as before, $C = 2py$, $x = 2hp$, and $H = Ay$, it will be $1{,}5708\,DD + 0{,}7854\,CC - 0{,}31416\,xx : \times \frac{1}{3}H =$ the Frustum $SApy$. And if we make $L = 2H$,

Then $1{,}5708\,DD + 0{,}7854\,CC - 0{,}31416\,xx : \times \frac{1}{3}L =$ the Double of that Frustum, being the Middle Zone. And by turning these Factors into one common Divisor, as in the Frustum of the Conoid at *Theorem* 25. *Page* 430. there will arise this following *Theorem*.

Theorem XXVII, $\left\{ 3{,}8196)\ 2DD + CC - 0.4xx : \times L \ (= \right.$ the Middle Zone of a Parabolic Spindle.

It may be expected, that I should now proceed to shew how the Area of any Hyperbola, and the Contents of such Solids as may be form'd by the Rotation of that Figure about its Axis, &c. may be found. But because those Things cannot be exactly perform'd by any certain or settled *Theorems*, as these of the Circle, Ellipsis and Parabola have been, I have therefore omitted them, and refer the Reader to Dr. *Wallis*'s *Algebra*. *Chap.* 90, &c. or to the *Philosoph. Transact.* Number 34. wherein he may find the Method of forming infinite Series relating to the squaring of any Hyperbola, &c. which are too tedious to be fully explain'd and demonstrated in this small Tract, it being only intended as an Introduction, which I shall here conclude.

APPEN-

AN APPENDIX OF Practical Gaging.

THE Art of Gaging is that Branch of the Mathematicks call'd *Stereometry*, or the *Measuring* of *Solids*, because the Capacities or Contents of all Sorts of Veſſels uſed for Liquors, *&c.* are computed as tho' they were really Solid Bodies, which any one that hath made himſelf Maſter of the foregoing Parts of this Treatiſe, may eaſily underſtand, without any further Directions.

However, becauſe it is not to be ſuppos'd, that every one who deſigns to undertake the Office or Employment of a Gager, hath made ſo great a Progreſs in Mathematical Learning, I have therefore preſented the young Gager with this Appendix, wherein I have only inſerted ſuch Rules as are uſeful in Gaging, and have been already demonſtrated in this Treatiſe. But herein, I preſuppoſe that he hath acquir'd, (or if not, it is very requiſite he ſhould acquire) a competent Knowledge both in Arithmetick and Geometry. That is.

I. In Arithmetick he ſhould underſtand the principal Rules very well, eſpecially Multiplication and Diviſion, both in whole Numbers and Decimal Parts; (which may be eaſily learn'd out of the 2d, 3d, and 5th Chapters of *Part* 1.) that ſo he may be ready in computing the Contents of any Veſſel, and caſting up his Gages by the Pen only, *viz.* without the Help of thoſe Lines of Numbers upon ſliding Rules, ſo much applauded, and but too much practis'd, which at beſt do but help to gueſs at the Truth. I mean ſuch Pocket Rules as are but nine Inches, or a Foot long, whoſe Radius of the double Line of Numbers is not ſix Inches; and therefore the Graduations or Diviſions of thoſe Lines are ſo very cloſe, that they cannot be well diſtinguiſh'd. 'Tis true, when the Rules are made two or three Foot long, (I had one of ſix Foot) then they may be of ſome Uſe, eſpecially in ſmall Numbers; altho' even then, the Operations may be much better, (and almoſt as ſoon) done by the Pen; for indeed the chief Uſe of ſliding Rules, is only in taking of Dimenſions, and for that Purpoſe they are very convenient.

II. In Geometry the Gager ſhould underſtand not only how to take Dimenſions, (which is beſt learn'd by Practice) but alſo how to divide any irregular Figure or Superficies, as Brewers Backs, or Coolers, &c. into the eaſieſt and feweſt Regular Figures they will admit of, that ſo their Area's may be truly computed with the leaſt Trouble. And this may be learn'd (with a little Care and Diligence, out of the 1ſt, 2d, and 5th Chapters of *Part* III. which the Gager ſhould be well acquainted with.

Alſo he ought to have ſo much Skill in Solids, as to be able, even at Sight (but this muſt be acquir'd by Experience) to determine what Sort of Figure any Veſſel is of, (*viz.* any Tun, or cloſe Cask) or what Figure it may be beſt reduc'd to, ſo that its Dimenſions may be truly taken, and the Content thereof computed with the leaſt Error. I ſay, with the leaſt Error, becauſe it is very difficult, if not impoſſible, to do it exactly; for there is not any Tun or Cask, &c. ſo regularly made, as by the Rules of Art it is requir'd to be.

III. Beſides the aforemention'd, the young Gager muſt know, that all Dimenſions uſeful in Gaging, are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Meaſures, as Feet, or Yards, &c. thoſe Meaſures muſt be reduc'd to Inches (ſee *Sect.* 4. *Page* 42.) becauſe the Contents of all Sorts of Veſſels taken Notice of in Gaging) are computed by the Standard Gallon of its kind, whoſe Content is known to be a certain Number of Cubic Inches. That is, the Beer or Ale Gallon contains 282, the Wine Gallon 231, and the Corn Gallon, 268,8 Cubic Inches. (See the five Tables, &c. in *Page* 34. 35, 36, which I here ſuppoſe the Gager to have learn'd perfectly by Heart.) Conſequently, if either the Superficial or Solid Content of any Veſſel, as Back, Tun, Cask, &c. be once computed in Cubic Inches, it will be eaſy to know how many Gallons, either of Ale, Wine, or Corn, that Veſſel will hold.

Note, I have here ſaid, the Superficial Content in Cubic Inches, which may ſeem to be very improper, according to the Definition given of a Superficies in *Page* 279. But you muſt know, that in the Buſineſs of Gaging all Superficies, or Area's, are always underſtood to be one Inch deep; otherwiſe it could not be ſaid (as in the Gager's Language it is) that the Area of ſuch a Back, or of ſuch a Circle, &c. is ſo many Gallons.

Theſe Things being very well underſtood, the young Gager will be fitly prepar'd to underſtand the following Problems, which are ſuch as have (moſt of them) been already propos'd in the foregoing Parts of this Treatiſe, and only are here apply'd to Practice; and therefore I ſhall, for Brevity's ſake, often refer to thoſe Theorems and Problems.

Sect. 1. *To find the Area of any Right-lin'd Superficies in Gallons.*

PROBLEM I.

To find the Area of any Square Tun, Back, or Cooler, &c. *either in Ale, Wine, or Corn Gallons.*

Rule. *Multiply the given Length or Breadth* (being here equal) *into itself, and the Product will be the Area in Inches; then divide that Area by* 282, *or* 231, *or* 268,8 *and the Quotient will be the Area requir'd.*

Example, Suppose the Side of a Square Tun, Back, or Cooler, be 124,5 Inches, what will its Area be in Gallons?

First 124,5 × 124,5 = 15500,25 the Area in Inches,
Then 282) 15500,25 (*54,96, &c.* the Area in Ale Gallons.
And 231) 15500,25 (*67,10, &c.* the Area in Wine Gallons.
Or 268,8) 15500,25 (*57,66, &c.* the Area in Corn Gallons.

But if any one would rather work by Multiplication than by Division, he may turn or change any Divisor into a Multiplicator, if he divide Unity, or 1, by that Divisor, (vide *Problem* III. *Page* 402.)

Thus 282) 1,000000 (0,003546 the Multiplicator for Ale Gallons.
And 231) 1,000000 (0,004329 the Multiplicat. for W. Gallons.
Or 268,8) 1,000000 (0,003722 the Multiplicat. for Corn Gallons.

Consequently, 15500,25 × 0,003546 = 54.96, *&c.* the Area in Ale Gallons. As before; and so on for the rest.

PROBLEM II.

To find the Area of any Tun, Back, or Cooler, in the Form of a Right-angled Parallelogram, in Ale Gallons, &c.

See the Rule for finding its Area in Inches, as *Prob.* 1. *Page* 339. Then either divide (or multiply) that Area, as above, and you will have the Area in Gallons.

Example, Suppose the Length of a Brewer's Tun, Back, or Cooler, be 217,5 Inches, and its Breadth 85,6 Inches, what will its Area be in Ale or Beer Gallons, *&c.*

First 217.5 × 85,6 = 18648. Then 282) 18648 (*66.12, &c.*
Or 18648 × 0,003546 = *66.12, &c.* the Area requir'd, *&c.*

PROBLEM IV.

To find the Area of any Triangular Tun, Back, or Cooler, in Ale Gallons, &c.

See the Rule for finding its Area in Inches, at *Prob.* 3. *Page* 340. then divide (or multiply) that Area as before, and you will have the Area requir'd.

Example, If the Length of the Base of a Triangular Cooler be 86,4 Inches, and its Perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

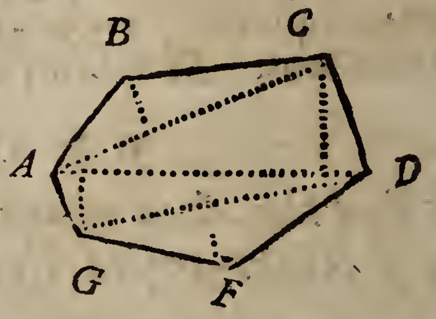
First, $86{,}4 \times \frac{57}{2} = 2462{,}4$. Then $282)\ 2462.4\ (8.73$, &c.

Or $2462.4 \times 0{,}003546 = 8.73$, &c. the Area in Ale Gallons.

Proceeding thus, you may easily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or of any other Polygon, either Regular, or Irregular, in Ale or Beer Gallons, &c. if you first divide it into Triangles, and then find the Area's of those Triangles, (as in the 2d, 4th, 5th, and 6th Problems in Chapter 5. Part 3.) the Sum of those Area's being divided (or multiply'd) by its proper Divisor, (or Multiplicator) as above, will give the Area requir'd.

Now the practical Way of dividing any Polygonous Tun, Back, &c. into Triangles, is by help of a chalk'd Line, such as the Carpenters use, and may be thus perform'd.

Suppose any Brewer's Tun, Back, or Cooler, in the Form of the annex'd Figure *ABCDFG*. Let one End of the chalk'd Line be fasten'd with a Nail (or otherwise) in any Corner or Angle of the Back, as at *A*, then straining it to the Angle at *C*, strike the Diagonal Line *A C*, upon the Bottom of the Back, and straining it again to the Angle *D*, strike another Diagonal Line, as *AD*, and so on for the Diagonal Line *GD*, &c. Then having mark'd out all the Diagonals, the Perpendiculars may be thus found:



Fasten (as before) one End of the chalk'd Line in the Angle *B*, and then by moving it to and fro upon the Stretch, find out the nearest Distance between the Angle at *B*. and the Diagonal Line *AC*; there strike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*. and so on for the other Perpendiculars, which being all mark'd out upon the Bottom of the Back, measure them, and each Diagonal by a Line of Inches, &c. And then the Area of that Back may be computed, as directed above.

And

And here, by the way, it may be obſerv'd, That the Number of Triangles will always be leſs by two; and the Number of the Diagonals leſs by three, than the Number of the Sides of any Right-lin'd Figure that is ſo divided.

Having found (as above) the true Area of any Brewer's Back or Cooler, (which, according to the Laws of Exciſe, ought always to be fix'd or immovable) the next Thing will be to find out the true Dipping or Gaging Place in that Back, that ſo the true Quantity of Worts may be computed or caſt up at any Depth; which may be thus done.

1. When the Bottom of the Back is cover'd all over (of any Depth) either with Wort or Liquor, (*viz.* Water) then dip it in eight or ten ſeveral Places (more or leſs according to the Largeneſs of the Back) as remote and equally diſtant one from another as you well can, noting down the wet Inches and Decimal Parts of every Dip.

2. Divide the Sum of all thoſe Dips or wet Inches by the Number of Places you dipp'd in, and the Quotient will be the Mean Wet of all thoſe Dips.

3. Laſtly, find out ſuch a Place by the Side of the Back (if you can) that juſt wets the ſame with that mean Dip, and make a Notch or Mark there for the true and conſtant dipping Place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipp'd or gaged at that Place, and the wet Inches ſo taken be multiply'd into the Area of the Back in Gallons, the Product will ſhow what Quantity (*viz*, how many Gallons) of Wort are in that Back at that Time, provided the Sides of the Back do ſtand at right Angles with its Bottom,

Sect. 2. *To find the Area of any Circular and Elliptical Superficies in Gallons.*

1. I have demonſtrated in Chapter 6. Part 3. and Theorem 3, 5, 6. Part 5. that the Periphery of the Circle, whoſe Diameter is Unity or 1. is 3,14159265, &c. (or for common Uſe 3,1416) And that its Area is 0,78539816, &c. (or 0,7854 *fere*)

2. Alſo, that the Peripheries of all Circles are in Proportion one to another, as the Diameters are; and their Area's are in Proportion to the Squares of their Diameters. That is,

As 1 : 3,1416 :: the Diameter of any Circle : To its Periphery.
And 1 : 0,7854 :: the Square of the Diameter : To the Area.

Upon

Upon these two Proportions depends the Solution of all the common or practical Questions about a Circle. See *Page* 408 and 409.

PROBLEM IV.

The Diameter of any Circle being given in Inches, to find the Periphery.

Rule. *Multiply the given Diameter with* 3,1416, *and the Product will be the Periphery requir'd.* See Prob. 1. Page 408.

Example, Suppose the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its Periphery.

Then 54,5 × 3,1416 = 171,21, *&c.* Inches, is the Periphery requir'd.

The Converse of this is easy, *viz.* by having the Periphery given, to find the Diameter. See *Prob.* 3. *Page* 408.

PROBLEM V.

The Diameter of any Circle being given, (in Inches) *to find its Area in Gallons.*

Rule. *Multiply the Square of the propos'd Diameter into* 0,7854, *and the Product will be the Area in Inches* (see Prob. 2. Page 408.) *that Area being divided by* 282 *or* 231, &c. *the Quotient will be the Area requir'd.*

Example, Suppose the given Diameter be 54,5 Inches, as above. First 54,5 × 54,5 = 2970,25. And 2970,25 × 0,7854 = 2332,83 the Area in Inches.

Then 282) 2332,83 (8,2724 the Area in Ale or Beer Gallons.
And 231) 2332,83 (10,0988 the Area in Wine Gallons.
Or 268,8) 2332,83 (8,6788 the Area in Corn Gallons.

But these Area's in Gallons may be much easier found, without knowing the Circle's Area in Inches as above, by having the Square of the Diameter of that Circle whose Area is one Gallon; which may be thus found, by *Theorem* 6. *Page* 407. 0.785398 : 1 :: 282 : 359,05 the Square of the Diameter of the Circle whose Area is 282 Cubic Inches, *viz.* one Ale Gallon.

And from this Proportion will arise these following Divisors;
Viz. 0,785398) 282,000000 (359,05 will be a Divisor for *A. G.*
And 0,785398) 231,000000 (294,12 will be a Divisor for *W. G.*
Or 0,785398) 268,800000 (342,24 will be a Divisor for *C. G.*

If

If the Square of the Diameter of any Circle be divided by any one of these constant or fix'd Divisors, the Quotient will shew that Circle's Area in their respective Gallons. As for Instance in the last Circle, whose Square of its Diameter is 2970,25.

Then 359,05) 2970,25 (8,2725 the Area in *A. G.*
And 294,12) 2970,25 (10,0988 the Area in *W. G.* } As before.
Or 342,24) 2970,25 (8,6788 the Area in *C. G.*

Now these Divisors may be turn'd into Multiplicators, by dividing Unity or 1. as in *Page* 435. Or rather by dividing the Area in Inches of that Circle whose Diameter is 1.

That is, 0,785398 by 282. Or by 231. *&c.*

Thus 282) 0,785398 (0,002785 the Multiplicator for Ale Gal.
And 231) 0,785398 (0,003399 the Multiplicator for Wine Gal.
Or 268,8) 0,785398 (0,002922 the Multiplicator for Corn Gal.

These Multiplicators are the respective Area's of a Circle whose Diameter is 1. And therefore, if the Square of the Diameter of any Circle be multiply'd with any of these Numbers, the Product will be that Circle's Area in Gallons of the same Name.

Viz. $2970{,}25 \times 0{,}002785 = 8{,}2725$ the Area in Ale Gal. as above.
And $2970{,}25 \times 0{,}003399 = 10{,}0988$ the Area in Wine Gallons, *&c.*

Thus you see, that if the Diameter of any Circle be given in Inches, there are three several Ways of finding its Area in Gallons, and all equally true; but that which is perform'd by the constant Divisors is most generally practis'd.

PROBLEM VI.

The Transverse (or longest Diameter) *and the Conjugate* (or shortest Diameter) *of any Elliptical Superficies being given, to find its Area in Gallons.*

Rule. { *Multiply the two Diameters,* (*viz. the* Length *and* Breadth) *together, and divide their Product by* 359,05 *for Ale Gallons, or* 294,12 *for Wine Gallons,* &c. *the Quotient will be the Area requir'd.* See *Theorem* 7. *Page* 412.

Example. Suppose the longest Diameter to be 73,5 Inches, and the shortest Diameter to be 51,6 Inches, what will the Area be in Ale Gallons?

First $73{,}5 \times 51{,}6 = 3792{,}6$. Then 359,05) 3792,6 (10,56 the Area in Ale Gallons. Or 294,12) 3792,6 (12,89 the Area in Wine Gallons, *&c.*

Now,

Note, The two laſt Problems are of great Uſe in Gaging of Wort amongſt Country Victuallers, who generally brew but ſhort Lengths of Ale, (*viz.* perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in ſeveral ſmall open Veſſels or Tubs, whoſe Baſes or Bottoms are either a Circle, or an Ellipſis, having their Sides but low, and are moſt commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Wort that is at any Time in one of thoſe open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom, according to its Figure, (as above) and either mark that Area on the outſide of the Tub, which was the Way I generally us'd to order, because the Victuallers did often lend their cooling Tubs one to another) or elſe number the Tub, and enter its Area (and its Number) into the Stock-Book; then when any of thoſe Tubs hath Wort in it, take the Diameter of the Surface or Top of the Wort, and find that Area, adding it and the Bottom Area together. If either the half Sum of thoſe two Area's be multiply'd with the Depth of the Wort, (taken as near the Middle of the Tub as you well can) or, if the Sum of thoſe two Area's be multiply'd with half the Depth, (ſo taken) the Product will ſhew the Quantity of the Wort very near the Truth.

PROBLEM VII.

The Diameter of any Circle, and the verſed Sine (*viz.* the Height) *of any Segment being given, to find the Area of that Segment in Gallons.*

In the 410th and 412th Pages, you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231, *&c.* the Quotient will be its Area in Gallons. But becauſe the Area of any ſuch Segment may be readily found in Gallons (without finding its Area in Inches) by help of a Table of Segments, whoſe Conſtruction is laid down in the Problem, *Page* 411, *&c.* I have here inſerted a Compendium of ſuch a Table, which will ſerve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons; but alſo to find the Number of Gallons that are either drawn out, or remaining in any Cylindric Veſſel lying along; or of any cloſe Cask (being firſt reduc'd to a Cylinder) its Axis lying parallel to the Horizon, uſually call'd the Ullage of a Cask, as ſhall be ſhew'd further on.

A Table of the Segments of a Circle, whose Area is Unity or 1. The Diameter being divided by parallel Chord-Lines into 100 equal Parts.

V. S.	Segment.	V. S.	Segment	V. S.	Segment.	V. S.	Segment.
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,7002	1	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

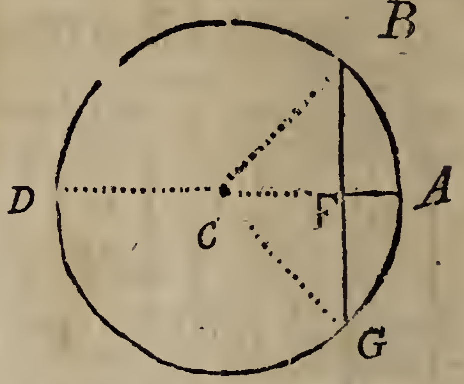
The Use of this Table of Segments depends upon the following Proportion:

Viz. { As the Diameter of any proposed Circle : is to 100 : (the Diameter of the Tabular Circle) : : so is the Height of any Segment of the propos'd Circle : to a Versed-Sine in the Table.

Then if the Tabular Segment, which stands against that Versed-Sine, be multiply'd into the Circle's Area, (either in Inches or Gallons) the Product will be the Area of the Segment requir'd, (of the same Name) *viz.* If the Circle's Area be Inches, the Segment will be Inches: If Gallons, the Segment will be Gallons.

 Example,

Example. Let the Diameter of the given Circle be $DA = 62{,}5$ Inches, and the Height of the Segment fought, be $FA = 20$ Inches, what will its Area be in Ale-Gallons?



First, the Area of the whole Circle will be 10,8793 Ale-Gallons. By *Problem* 5. And the Proportion will ſtand thus, $62{,}5 : 100 :: 20 : 32$ the Verſed-Sine of the Table, whoſe Segment is 0,2759.

Then $10{,}8793 \times 0{,}2759 = 3{,}0016$ Ale-Gallons, being the Area of the Segment *B A G F*, as was requir'd. The like may be done for Wine-Gallons, Corn-Gallons, or Inches.

And upon Occaſion, the like Segments of any Ellipſis may be eaſily found. See the Proportions in the Corollaries to the 7th and 8th Theorems, *Page* 412, *&c.* to which I here for Brevity's ſake refer the Reader.

Sect. 3. *To compute the* Contents *of ſuch Veſſels* (viz. *Tuns*, &c.) *as are in the Form of the following Solids.*

Note, Before the young Gager proceeds to theſe Computations, he ſhould be well acquainted with ſuch Solids as are defin'd in *Page* 402 and 403. And then he may eaſily underſtand what Sort of Figures are meant in the following *Problems*, without the Repetition of many Words.

PROBLEM VIII.

To find the Content *of any Priſm, whoſe Sides are parallelograms; what Form ſoever its Baſe is of.*

That is, to compute the Content (in Gallons) of any Tun, *&c.* whoſe Sides are Parallelograms which ſtand upright, or at right Angles with its Bottom.

Firſt, find its ſolid Content in Inches by Theorem 9. *Page* 414. Then divide that Content by 282. or 231. Or by 268,8 the Quotient will ſhew the Content in their reſpective Gallons, *viz.* in Ale, Wine, or Corn-Gallons.

Or elſe multiply the Content in Inches with 0,003546, (Or 0,004329, *&c.* (See the Multiplicators, *Page* 435.) thoſe Products will be the Content in their reſpective Gallons.

Or otherwiſe thus.

Find the true Area of the Tun's Baſe or Bottom, as directed in *Section* 1, *Page* 435. That Area being multiply'd with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, as before.

I take

I take the Work of this Problem to be so very easy, it needs no Example.

PROBLEM IX.

To find the Content *of any Pyramid* (in Gallons) *whose Base is bounded with Right-lines.*

Every Pyramid is one third Part of its circumscribing Prism, by Theorem 10, *Page* 415. Therefore

If the Area of the Base of any Pyramid in Gallons, be multiply'd into one third of its perpendicular Height; or if one third of that Area be multiply'd with the whole Height, either of those Products will be the Content of the Pyramid in Gallons, *&c.*

But the Content of any Square Pyramid may be easily found in Gallons by this Rule.

Rule. { *Square the Side of its Base, and multiply that Square with the perpendicular Height; then divide that Product by* 846 = 282 × 3 *for Ale-Gallons, or by* 693 = 231 × 3 *for Wine-Gallons, or by* 806,4 = 268,8 × 3 *for Corn-Gallons, the Quotient will be the* Content *requir'd.*

Or if you multiply the said Product with 0.001182 for A. G.
Or with 0.001443 for W. G. Or lastly with 0,001241 for C. G. the Result will be the Content requir'd, As before.

PROBLEM X.

To find the Content (in Gallons) *of the Frustum of any square Pyramid, cut off by a plane Parallel to its Base.*

First. Either by Theorem 15. *Page* 419. or *Theorem* 16. *Page* 420, find the propos'd Frustum's Solidity in Cubic Inches. Then divide that Content in Cubic Inches by 282: or 231, *&c.* and the Quotient will be the Content of the Frustum in their respective Gallons.

But from the aforesaid Theorem 15. there may be easily deduc'd the following general Rule for finding the Content of the like Frustum of any Pyramid what Form soever its Bases are of (supposing them to be parallel) whether they are alike or unlike.

Rule. { *First find the Area of each Base* (viz. *the Top and Bottom Area's of the propos'd Frustum*) *then find a geometrical Mean between those two Area's,* (*by* Lemma 1. Page 83.) *the Sum of those two Area's, and their Mean, being multiply'd into one Third of the Frustum's Height, will produce the* Content *requir'd.*

Example, Suppose a Tun in the Form of the lower Frustum of a Pyramid, whose Bases are equilateral Triangles. Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height (*viz.* Depth) be 33 Inches, what will the Content of that Tun be in Ale-Gallons?

First find the Area of each Base in Inches, by Problem 7. *Page* 343; then find what those Area's are in Ale-Gallons, by Problem 3. *Page* 436. Multiply those two Area's together; the square Root of their Product will be the Mean Area, *&c.* as in this *Example.*

Example.	The Area of the Top is	2,71	Ale-Gallons.
	The Area of the Bottom is	6,12	
	The Mean Area will be	4,07	
	Their Sum is	12,90	

Then $12,9 \times \frac{33}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the Content requir'd.

PROBLEM XI.

To find the Content *of any Right* Cylinder *in* Gallons.

That is, to compute the Content of any round Tun, *&c.* whose Diameters at Top, and Bottom are equal, and at Right-angles with its Sides.

The Content of such a Tun may be found by Theorem 11. *Page* 415. Or otherwise by the following Rule.

Rule. *Multiply the Square of the Diameter into the Height and divide the Product by* 359,05 (*or multiply with* 0,002785) &c. *as in* Page 439. *that Quotient or Product will be the* Content *requir'd.*

Exam. Suppose the Diameter be 42,5 and the Height 31,5 Inches.
First $42,5 \times 42,5 = 1806,25$ And $1806,25 \times 31,5 = 56896,875$
Then 359,05) 56896,875 (158,46 the Content in A. Gal. *&c.*

PROBLEM XII.

To find the Content *of any Cone or round Pyramid in Gallons.*

Because every Cone is one Third of its circumscribing Cylinder, (See Theorem 13. *Page* 416.) therefore its Content may be truly found by the following Rule.

Rule. *Multiply the Square of the Diameter of its Base, into the perpendicular Height; then divide their Product by* $1077,15 = 359,05 \times 3$ *for Ale-Gallons, or by* $882,36 = 294,12 \times 3$ *for Wine-Gallons, and the Quotient will be the* Content *requir'd.*

Or

Or if the said Product be multiply'd with $0,000928 = \frac{0,002785}{3}$ or with $0,001133 = \frac{0,0034}{3}$ those Products will be the Content in their respective Gallons.

Example, Suppose the Diameter of the Base be 42,5. and the perpendicular Height be 31,5 Inches, what will the Content be in Ale-Gallons? As before.

First $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$

Then 1077,15) 56896,875 (52,82. Or $56896,25 \times 0,000928 = 52,82$ the Content in Ale-Gallons. And so on for Wine or Corn-Gallons.

PROBLEM XIII.

To find the Content *of the Lower Frustum of any* Cone, *in Gallons.*

That is, to compute the Content of any round Tun, *&c.* whose Diameters at Top and Bottom are parallel but unequal.

The Content of such a Tun may be found by the Rule at Problem 10. but from Theorem 16. *Page* 420. it will be easy to deduce this following Rule.

Rule. *To the triple Product of the Top and Bottom Diameters, add the Square of their Difference; Multiply that Sum into the Height (or Depth) then divide the last Product by* 1077,15 *for Ale-Gallons, or by* 882,36 *for Wine-Gallons, the Quotient will be the* Content *requir'd.*

Example, Suppose the Diameter at the Top be 52,4 Inches, the Diameter at the Bottom 44,6, and the Height 30 Inches.

First, $52,4 \times 44,6 = 2337,04$. And $2337,04 \times 3 = 7011,12$ } Add
Also $52,4 - 44,6 = 7,8$. And $7,8 \times 7,8 = 60,84$ }

The Height $30 \times 7071,96 = 212158,8$

Then 1077,15) 212158,80 (196,96 } the Content in A. Gal.
Or $212158 \times 0,000928 = 196,96$ }

And so on for either Wine or Corn-Gallons, as Occasion requires. But if the Tun (or Vessel) be not truly circular, that is, if either its Top or Bottom (or both of them) be elliptical, whether they are alike, or unlike it matters not. the Content of such a Tun may be truly found by the General Rule at *Problem* 10.

PROBLEM XIV.

The Axis or Diameter of any Sphere or Globe, being given (in Inches) to find its Content *in Gallons.*

Every Sphere is two Thirds of its circumscribing Cylinder. by Theor. 18. *Page* 423. from whence and Theor. 20. *Page* 426. it is prov'd,

prov'd, that if the Cube of the Axis of any Sphere (taken in Inches be multiply'd into 0,5236. the Product will be the Content of that Sphere in Cubic Inches. Consequently, if that Content be divided by 282, or by 231, *&c.* the Quotient will be the Content in Gallons.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, *&c.* may be contracted into one.

Thus 282) 0,5236 (0,001856 will be a Multiplicat. for A. G.
And 231) 0,5236 (0,002266 will be a Multiplicat. for W. G.
Or 0,5236) 282 (538,57 will be a Divisor for Ale-Gal.
And 0,5236) 231 (441,17 will be a Divisor for Wine-Gal.

From hence arises this following Rule.

Rule. *If the Cube of the Axis of any Sphere, be divided by* 538,57 (or multiply'd with 0,001856) *or divided by* 441,17, (or else multiply'd with 0,002266) *the Quotient* (or Product) *will be the Sphere's Content in their respective* Gallons.

Example, Suppose the Axis or Diameter of a Sphere or Globe, be 22 Inches, how many Ale-Gallons may it hold?
Then 22 × 22 × 22 = 10648. And 538,57) 10648 (19,76 A. G.
Or 10648 × 0,001856 = 19 76 Ale-Gal. the Content requir'd.
And so, for either Wine, or Corn-Gallons as Occasion requires.

PROBLEM XV.

To find the Content *of any Segment of a Sphere in* Gallons.

In the Scholium, *Page* 424, there are two Theorems for resolving of this Problem according to the *Data*.

1. If the Diameter of the Segment's Base, and its Height are given, the Content may be found by the first of those Theorems, which gives this Rule.

Rule 1. *To the triple Square of Half the Diameter add the* Square *of the Height; then multiply that* Sum *into the Height, and divide the Product by* 538,57 *for A. G. or by* 441,17 *for W. G.* &c. *As above.*

2. But if the Axis of the Sphere, and the Height of the Segment are given, the Content may be found by the second of those Theorems.

Rule 2. *From the* triple Product *of the Axis into the Height, subtract twice the* Square *of the Height; then* multiply *the Remainder into the Height, and divide that Product by* 538,57, &c. *As in the* last Problem.

Either

Either of these Rules will produce the Content of the Segment in Gallons.

Example. Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches, what may it contain in Ale-Gallons?

First, 2) 28 (14. Then (by Rule 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly, 538,57) 3744 (6,95 the Content requir'd.

Note, This Problem may be of Use in Gaging the Crowns of Brewers Coppers, *&c.*

Sect. 4. *The* practical Method *of Gaging any fix'd Tun or Copper, and making a* Table *to shew what it will hold at every Inch Deep, usually call'd Inching of a* Tun, *&c.*

First, you must know, that most (if not all) Brewers Tuns are so fix'd, as to lean a little for Conveniency of cleansing their Drink, which is usually call'd the Drip or Fall of the Tun. Now this Drip or Fall of any Tun, is the Hoof of such a Solid as that Tun is suppos'd to represent; and under that Consideration it may be found, as in Theorem 16. *Page* 420. But the practical (and indeed the best) Way, is to measure into the Tun (when it is dry) so much Liquor as will just cover its Bottom; for by that Means you do not only find the true Fall, but also a true horizontal or level Plane over the Bottom of the Tun; from which, if the Depth of the Tun (*viz.* the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true parallel Plane at the Top of the Tun to that of the Liquor.

Then if the Sides of the Tun are streight, from the Top to the Bottom, take as many Dimensions in the aforesaid two Planes, as are needful to find the true Area of each; and by those two Area's, and the aforesaid Depth, find so much of the Tun's Content (by the general Rule at Problem 10.) as is betwixt those two Planes.

Next, to Inch that Tun, divide the Difference between the Top and Bottom Area's by the aforesaid Depth, and the Quotient will be an Addend or fix'd Number; which being added to the lesser Area, the Sum will be the Area of the next Inch: and being added to that Area, their Sum will be the Area of the third Inch, and so on from Inch to Inch, until the Area of every single Inch be found, the Sum of those Area's (if the Work be true) will amount (or be equal) to the Content found, as above. And if the

the Tun's Drip or Fall be added to the Sum of all thoſe Area's, that Sum will be the whole or full Content of that Tun.

Now from hence it muſt needs be eaſy to conceive, that if 1. 2. 3. or any Number of thoſe Area's accounted from the Bottom, be added to the Fall, that Sum will ſhew the Quantity of Liquor or Drink that is in the Tun, to ſuch a Number of wet Inches from the Bottom, as there were Area's added together.

Or if the Sum of any Number of thoſe Area's (being accounted from the Top) be ſubtracted from the Tun's whole Content, the Remainder will ſhew what Quantity of Liquor or Drink is in the Tun, when there is ſuch a Number of dry Inches from the Top as there were Area's ſubtracted.

This being well conſider'd, it will be eaſy to make a Table, either to every wet or dry Inch of any regular Tun, (*viz.* whoſe Sides are ſtrait from Top to Bottom) what Form ſoever its Baſes are of; and whether it ſtand upon the greater, or leſſer Baſes.

But if the Sides of the Tun are irregular (*viz.* not ſtrait from its Top to the Bottom) then the beſt and eaſieſt Way will be to divide or part the Tun into ſeveral Fruſtums, each of 10 Inches deep; and finding the Content of every ſingle Fruſtum, by taking the Diameters in the Middle of every one of thoſe 10 Inches, (that is, the firſt Diameter, at 5 Inches from the Top; the ſecond Diameter at 15 Inches from the Top, *&c,*) and multiplying their reſpective Area's with 10, (which is done by only removing the ſeparating Comma's one Place forward to the Right-hand) if the Sum of all thoſe Fruſtums be added to the Fall, (as before) that Sum will be the whole Content of the Tun.

Note, If you take the Height of the aforeſaid 10 Inch Fruſtums in the Side of the Tun, you muſt allow for the Difference between the ſlant Height and the perpendicular Height in every Fruſtum.

Laſtly, if from the whole Content of the Tun you ſubtract the mean Area of the firſt Fruſtum ten times, and from the Remainder ſubtract the mean Area of the ſecond Fruſtum ten times, and from the laſt Remainder ſubtract the mean Area of the third Fruſtum. *&c.* until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will ſhew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is alſo the Method of Gaging and Inching of Brewers Coppers, *viz.* by firſt meaſuring into the Copper ſo much Liquor as will juſt cover its Crown; and then dividing its perpendicular Height into Fruſtums, and its Sides into four equal Parts, that ſo the croſs Diameters may be taken in the Middle of

each

each Frustum: But if the Copper be much wider at the Top than at the Bottom, and its Sides spheroidal or arching, as generally all large Coppers are; then, instead of taking those mean Diameters in the Middle of every ten Inches, as above, you must take them in the Middle of every six Inches, and proceed on as before.

Now the Quantity of Liquor that would cover the Crown of the Copper, may be found without measuring it, as above. In order to that, I do suppose the Crown to be the Segment of a Sphere, and the lower Part of the Copper, wherein the Crown ariseth, to be the Frustum of a parabolic Conoïd; then if the Diameter at the Top of the Crown, and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule.

Rule. *From the Area of the Plane at the Top of the Crown, subtract* $1\frac{1}{3}$ *of the Area of the Crown's Height; the Remainder being multiply'd into Half the Height of the Crown will produce the Quantity or Number of Gallons that will cover the Crown.*

This Rule is deduc'd from *Scholium, Page* 424, and *Theorem* 15. *Page* 430.

Sect. 5. *To compute the* Content *of any* close Cask *in Gallons*, viz. *of any Butt, Pipe, Hogshead, Barrel, &c.*

In order to perform this difficult Part of Gaging, the three following Dimensions of the propos'd Cask must be truly taken in Inches, and Decimal Parts of an Inch.

Viz.
- The Bulge or Bung Diameter within the Cask.
- Either of the Head Diameters, supposing them both equal.
- And the Length of the Cask within.

Note, *In taking of these Dimensions, it must be carefully observ'd,*

1. That the Bung-hole be in the Middle of the Cask; also, that the Bung-staff, and the Staff over-against the Bung-hole, are both regular or even within.

2. That the Heads of the Cask are equal and truly circular; if so, the Distance between the Inside of the Chine to the Outside of its opposite Staff, will be the Head Diameter within the Cask, very near.

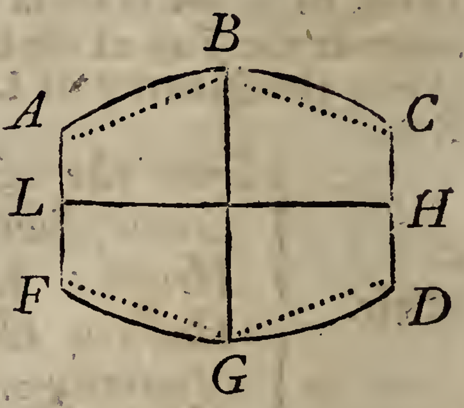
3. With a sliding Pair of Calipers (made on purpose for that Use) take the shortest Distance or Length between the Outsides of the two Heads; (supposing them even) from that Length subtract $1\frac{1}{2}$ Inch (more, or less, according to the Largeness of the

 Cask)

Cask) for the Thickneſs of the two Heads, the Remainder will be the Length of the Cask within.

Now, by theſe Dimenſions, one would ſuppoſe the Content of the Cask were perfectly limited; but it will be eaſy to perceive, by the following Figure, that the Diameters (aboveſaid) and the Length of one Cask may be equal to thoſe of another, and yet one of thoſe Casks may contain or hold ſeveral Gallons more than the other.

As for Inſtance, ſuppoſe the annexed Figure *ABCDGF*, to repreſent a Cask; then it is plain, that if the outward curved Lines *ABC*, and *FGD* are the Bounds or Staves of the Cask, it muſt needs hold more than if the inner ſtrait or prick'd Lines were its Bounds or Staves; and yet the Bung Diameter *BG*, Head Diameter *CD* and *AF*, and the Length *LH* are the ſame in both thoſe Casks.



Whence it plainly appears, that no one certain or general Rule can be preſcrib'd to find the true Content of all Sorts of Casks, and therefore Gagers do uſually ſuppoſe every Cask to be in the Form of ſome one of theſe following Solids.

Viz.
- I. The middle Zone or Fruſtum of a Spheroid.
- II. The middle Zone or Fruſtum of a Parabolic Spindle.
- III. The lower Fruſtums of two equal Parabolic Conoids.
- IV. The lower Fruſtums of two equal Cones.

Now the Way of gueſſing at the Cask's Form, and computing its Content according to that ſuppos'd Form, I ſhall here ſhew in their Order.

I. If the Staves of the Cask are very curved or arching (as the outward Lines of the laſt Figure) then the Cask is ſuppos'd to be in the Form of the middle Zone or Fruſtum of a Spheroid, whoſe Content may be computed, by *Theorem* 22. *Page* 427. which gives theſe two Rules.

Rule I. *To twice the Square of the Bung Diameter, add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by* 1077,15. viz. 3,8197 × 282 *for Ale-Gallons; and by* 882,36. viz. 3,8197 × 231 *for Wine Gallons. Or thus*

Rule

Rule 2. { *To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their respective Gallons.*

Example 1. Suppose a Cask in the Form of the middle Zone of a Spheroid, whose Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 Inches.

First, $31,5 \times 31,5 \times 2 = 1984,5$. And $24,5 \times 24,5 = 600,25$
Again $1984,5 + 600,25 = 2584,75$. And $2584,75 \times 42 = 108559,5$.
Then $1077,15)\ 108559,5\ (100,78$ the Content in Ale-Gallons.
And $882,35)\ 108559,5\ (123,03$ the Content in W. Gallons.

Or thus, by the second Rule.

Bung Diameter, 31,5 twice its Circles Area is	5,5270
Head Diameter, 24,5 its Circles Area is	1,6718
The Length 42 divided by 3 is 14.	7,1988 = their *Sum*

Then $7,1988 \times 14 = 100,78$ the Content in Ale-Gal. As before.
And so the Content in Wine-Gal. may be found.

II. If the Staves of the Cask are not quite so much curved or arching as was suppos'd before, the Cask is then taken for the middle Frustum of a Parabolic Spindle, and its Content is computed, as by *Theorem* 27. *Page* 432. which gives this Rule.

Rule. { *To twice the Square of the Bung-Diameter, add the Square of the Head-Diameter; from their Difference subtract four Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by* 1077,15, &c. *As above.*

Example 2. Suppose the Dimensions the same as before. Then

$31,5 \times 31,5 \times 2 : + 24,5 \times 24,5 = 2584,75$. And $31,5 - 24,5 = 7$.
Again $7 \times 7 \times 0,4 = 19,6$. And $2584,75 - 19,6 : \times 42 = 107736,3$.
Then $1077,15)\ 107736,3\ (100,01$ the Cont. in A. G. *&c.* for W. G.

III. When the Staves of the Cask are but very little curved or arching then 'tis suppos'd to be in the Form of the Frustums of two equal Parabolic Conoids, abutting or joyning together upon one common Base at the Bulge, and the Content may be found by *Theorem* 25. *Page* 430. which gives these Rules.

Thus. *To the Square of the Bung-Diameter add the Square of the Head-Diameter; multiply their Sum into the Length, and divide the Product by* 718,08 (Viz. $2{,}5464 \times 282$) *for Ale Gallons; or by* 588,22 (Viz. $2{,}5464 \times 231$) *for Wine Gallons. Or thus,*

Rule 2. *To the Area of the Bung-Circle add the Area of the Head-Circle; multiply the Sum into Half the Length, and the Product will be the Content requir'd.*

Example 3. With the ſame Dimenſions as before. Then
$31{,}5 \times 31{,}5 : + 24{,}5 \times 24{,}5 = 1592{,}5$. And $1592{,}5 \times 42 = 66885$
And 718,08) 66885 (93,01 the Content in Ale-Gallons.
Or 588,22) 66885 (113,7 the Content in Wine-Gallons.

IV. If the Staves of the Cask are ſtrait from the Bulge to the Head, as the inner prick'd Lines in the laſt Figure, (if ſuch a Cask can be made) it is then taken for the lower Fruſtums of two equal Cones, abutting or joining together upon one common Baſe at the Bulge. And its Content may be computed as at *Problem* 13. *Page* 445. or by *Theorem* 15. *Page* 419. Thus,

Rule *To the Sum of the Squares of the Head and Bung-Diameters add their Product; then multiply that Sum into the Length, and divide the laſt Product by* 1077,15. *Or by* 882,36. *The Quotient will be the Content,* &c.

Example 4. With the ſame Dimenſions as before.
Firſt, $31{,}5 \times 31{,}5 : + 24{,}5 \times 24{,}5 : + 31{,}5 \times 24{,}5 = 2364{,}25$
And $2364{,}25 \times 42 = 99298{,}5$ Then 1077,15) 99298,5 (92,18 the Content in Ale-Gallons. And ſo on for Wine-Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whoſe Forms all Casks are ſuppos'd to be. And by the Examples it appears, that four ſuch Casks as have their Dimenſions all equal, and the ſame with thoſe above-mentioned, their Contents will be as in the Margin.

	Ale-Gallons.	*Differ.*
I.	100,78	0,77
II.	100.01	7.00
III.	93,01	0,83
IV.	92,18	

From the Diſproportion or Inequality of theſe Differences, it will be eaſy to conceive, that there may be ſeveral Casks whoſe Contents cannot be truly found, according to the aforeſaid ſuppos'd Forms; and therefore, in order to rectify the ſaid Inequalities, ſome Authors (that have written upon this Subject) have laid down

down Theorems of their own Invention; and yet call'd them by theſe Names) others have propoſed Tables for the ſame Purpoſe. But ſince it is ſo, that we can only gueſs at the Truth, the plaineſt and eaſieſt Way is to be preferr'd in Practice; and that is, by finding ſuch a mean Diameter as will reduce the propos'd Caſk to a Cylinder. Thus,

Rule. *Multiply the Difference between the Head and Bung Diameters with 0,7 or with 0,65, or with 0,6 or with 0,55. according as the Staves of the Caſk are more or leſs arching; add the Product to the Head Diameter, and the Sum will be the mean Diameter requir'd. Then find the Content as at Prob.* 11. *Page* 444.

Example. With the ſame Dimenſions as before. Then the Bung Diameter leſs the Head Diam. is $31,5 - 24,5 = 7$. And

		M. D.		*A. G.*	*Cont.*	*Dif.*
24,5 +	7 × 0,7	= 29,4	its Area	2,4073 × 42	= 101,10	
	7 × 0,65	= 29,05	———	2,3504 × 42	= 98,71	2,39
	7 × 0,6	= 28,7	———	2,2941 × 42	= 96,35	2,36
	7 × 0,55	= 28,35	———	2,2385 × 42	= 94,03	2,32

From theſe it may be obſerv'd, that the Difference between each Caſk's Content is regular, and very near equal; which plainly ſhews, that there is not ſo much Room left for Error this Way of computing their Contents, as was by the aforeſaid Forms.

Now the firſt of theſe four (*viz.* with 0,7) is very commonly uſed amongſt Gagers for all Sorts of Caſks; but I did never gage any Caſk that would contain quite ſo much as that Rule did make it; and the Reaſon doth appear very plain from Theorem 22. *Page* 427. being compar'd with Theorem 19. *Page* 426. and the laſt Figure, *viz.* that no Caſk (being regularly made) can hold more than the middle Fruſtum of a Spheroid. But I always found by Experience, that if the ſecond and third of theſe Rules (*viz.* with 0,65 and 0,6) were duly apply'd, they would anſwer very near the Truth amongſt the common Sort of Caſks; and the fourth Rule (*viz.* with 0,55) will come pretty near the Truth in computing the Contents of Caſks, whoſe Staves are almoſt ſtrait betwixt the Head and the Bung, *viz.* ſuch as Wine-Pipes, *&c.*

Sect. 6. *To find what Quantity of Liquor is either drawn forth, or remaining in any ſpheroidal Caſk, uſually call'd the Ullage of a Caſk; which hath two Caſes.*

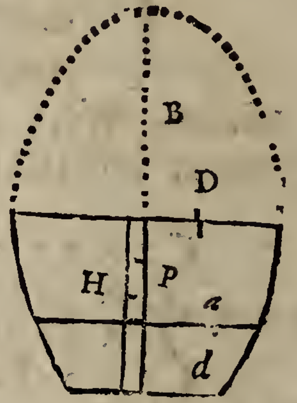
Caſe 1. *To find what Quantity of Liquor is in the Caſk, when its Axis is perpendicular to the Horizon,* viz. *when it ſtands upright upon one of its Heads.*

In

In order to perform this the easiest Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or Middle of the Cask is given. Now that may be done by this Proportion.

Viz. *As the Square of half the Length of the Cask : is to the Difference between the Bung and the Head Area's : : so is the Square of any Circle's Distance from the Bung : to the Difference between the Bung Area, and the Area of that Circle,* viz. *the Area of the Liquor's* Surface.

Demonstration.



Let $\begin{cases} H = \text{Half the Length of the Cask.} \\ D = \text{Half the Bung Diameter.} \\ d = \text{Half the Head Diameter.} \end{cases}$

And $\begin{cases} P = \text{the Distance of any Circle from the Bung.} \\ a = \text{Half the Diameter of that Circle.} \end{cases}$

Then, according to the common Property of the Ellipsis, *Page* 368. it will be,

$BB : DD :: BB - HH : dd$. And $BB : DD :: BB - PP : aa$.

Ergo $\dfrac{DDHH}{DD - dd} = BB$. And $\dfrac{DDPP}{DD - aa} = BB$.

Consequently, $\dfrac{DDHH}{DD - dd} = \dfrac{DDPP}{DD - aa}$

This Æquation being brought out of the Fractions, will become $DDHH - aaHH = DDPP - ddPP$.

Which gives this Analogy $HH : DD - dd :: PP : DD - aa$.

Then $DD - aa$ being subtracted from DD, will leave aa.

But Circles Area's are in Proportion to the Squares of their Diameters, by *Theorem* 6. *Page* 407. Therefore, *&c.* Q. E. D.

Then from the Bung-Area subtract one third Part of the aforesaid Difference, *viz.* between the Bung-Area, and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above or under Half the Content of the Cask.

Example. Let us suppose a Cask of the same Dimensions with that in the first *Example, Page* 451. and let it be requir'd to find what Quantity of Liquor is in it (of Ale-Measure) when there is but 9 Inches wet. Here Half the Length of the Cask is 21 Inches,

Inches, whose Square is 441, and the Liquor's Distance from the Bung is $21 - 9 = 12$. Its Square is 144. The Difference between the Bung and Head Area's is 1,0917 ($= 2,7635 - 1,6718$) Then 441 : 1,0917 :: 144 : 0,3564.

And $2,7635 - 0,3564 = 2,4071$ the Area of the Liquor's Surface.

Again 3) 0,3564 (0,1188. And $2,7635 - 0,1188 = 2,6447$

Then $2,6447 \times 12 = 31,7364$, what the Cask wants of being half-full. Consequently $50.39 - 31,73 = 18,66$ will be the Quantity of Liquor in the Cask at 9 Inches wet in Ale-Gallons.

And if the Cask had wanted but 9 Inches of being full; then $50.39 + 31.73 = 82,12$ would have been the Quantity of Liquor in the Cask.

Note, Because the two first Terms (*viz.* 441 and 1,0917) in the Proportion, are fix'd, *viz.* continue the same for any Distance, it will be very easy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will shew what Quantity of Liquor is either drawn out, or remaining in the Cask, at any Depth.

Case 2. *To find what Quantity of Liquor is in any* Cask, *when its Axis is parallel to the Horizon,* viz. *when it lies along.*

There are Variety of Tables to be found in Books of Gaging for this Purpose; but I always observ'd, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus perform'd:

1. By the Bung and Head Diameters, find such a mean Diameter as you judge will reduce the propos'd Cask to a Cylinder, by the Method laid down in *Page* 453. And then find its full Content, as in those Examples.

2. From the Bung Diameter subtract the mean Diameter, and halve their Difference, (*viz.* divide it by 2.)

3. From the wet-Inches of the propos'd Ullage, subtract the said half Difference, and call it x; then observe this Proportion.

Viz. { *As the Mean Diameter* : *is to* 100 (*the* Diameter of the Tabular Circle) :: *so is the last Difference* (viz. x) : *to a Versed*-Sine *in the* Table. (Page 441.

Then if the Tabular Segment which stands against that Versed-Sine, be multiply'd into the Content of the Cask, the Product will shew the Ullage, *viz.* what Quantity of Liquor is either in the Cask, or drawn forth.

Example,

Example 1. Let the Cask be that of the ſecond Sort, in *Page* 453. *viz.* whoſe Bung Diameter is 31,5 Inches, mean Diameter 29,05, and Content 98,71 Ale-Gallons; and ſuppoſe there were 10,5 Inches wet in it, it is requir'd to find the wet, and dry Gallons?

Here $31{,}5 - 29{,}05 = 2{,}45$ its half is 1,22. And $10{,}5 - 1{,}22 = 9{,}28$
Then $29{,}05 : 100 :: 9{,}28 : 0{,}319 =$ V. Sine; its Segment is 0,2748
And $98{,}71 \times 0{,}2748 = 27{,}12$ the Number of wet-Gallons.

Again $31{,}5 - 10{,}5 = 21$ the dry Inches; and $21 - 1{,}22 = 19{,}78$
Then $29{,}05 : 100 :: 19{,}78 : 0{,}68$; its Segment is 0,7241
And $98{,}71 \times 0{,}7241 = 71{,}48$ the Number of dry-Gallons.

Proof $71{,}48 + 27{,}12 = 98{,}6$ the Content of the Cask very near which plainly ſhews the Truth of this Method.

Thus far may ſuffice concerning Gaging of Backs or Coolers, Tuns, Coppers, and Casks, *&c.* To which I ſhall only add, That as the Contents of all Brewers Utenſils are to be computed by the Ale-Gallons, ſo the Contents of all Diſtillers Utenſils (*viz.* all their Waſh-Backs, Stills, and Casks, *&c.*) muſt be computed by the Wine-Gallons.

And in gaging of Malt, (upon which there is now a Duty of four Shillings *per* Buſhel) you muſt obſerve, That a Corn or Malt-Buſhel doth contain 2150,42 Cubic Inches; (See *Page* 42.) and therefore in Gaging of Malt-Ciſterns, or other Veſſels, 2150,42 will be a conſtant or fix'd Diviſor for finding the Area's of all Right-lin'd Figures in Buſhels at one Inch deep, and 2738 will be a conſtant or fix'd Diviſor for finding the Area's of Circular Figures.

I have omitted the Buſineſs of Gaging Maſh-Tuns, and taking an Account of the Goods or Grains, in order to eſtimate what Quantity of Wort was produced from them, *&c.* becauſe I could never find (by all my Obſervations) any Certainty therein; nor is it poſſible there ſhould be any, by reaſon of the great Difference that is in Malt; (and its Grinding too) for the beſt Malt (well ground) will yield or produce the moſt Wort, and leaſt Grains; on the contrary, bad Malt (being ill ground) yields the leaſt Wort and moſt Grains.

N. B. For the Numbers proper to the *IRISH* GALLON, See the laſt Edition of *HAWNEY*'s COMPLETE MEASURER, Publiſh'd by *S. FULLER*, at the *Globe* and *Scales* in *Meath-Street*, DUBLIN.

Thomas Dimmin's Book

A

SUPPLEMENT

Not in any of the former EDITIONS of this

BOOK.

Containing the

HISTORY

OF

Logarithms,

WITH

Several Easy METHODS of Constructing the *Tables* of the Logarithms and Sines, *&c.* Also the Demonstration of the Axioms and Doctrine of *Plane*

TRIGONOMETRY.

Extracted from the

Philosophical TRANSACTIONS and the WORKS of Dr. *KEIL*, *RONAYNE*, *WARD*, &c.

Cuncta Trigonus *habet, satagit quæ docta* Mathesis,
Ille *aperit clausum quicquid* Olympus *habet.*

THE PREFACE.

THE Mathematicks formerly received considerable Advantages; first, by the Introduction of the Indian Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phænomena of Nature; and lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Design therefore in the following Tract, is to supply what seemed still wanting, viz. to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.

The wonderful Invention of Logarithms we owe to the Lord Neper, who was the first that constructed and published a Canon

Canon thereof, at Edinburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The same Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr.* Henry Briggs, *Professor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the completing thereof: But the Lord* Neper *dying, the whole Business remaining was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compass'd a Logarithmic Canon, agreeable to that new Form for the first twenty Chiliads of Numbers, (or from* 1 *to* 20000) *and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Goudæ *in* Holland *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Brigg's *Prescriptions; but these Tables are not so useful as* Brigg's, *because the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themselves are called Natural. These Tables, together with their Construction and Use, were publish'd after* Brigg's *Death, at* London, *in the Year* 1633, *by* Heny Gellibrand, *and by him called* Trigonometria Britannica.

Since then, there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

The beſt Diſpoſition of theſe Tables, in my Opinion, is that, firſt thought of by Nathaniel Roe, *of* Suffolk; *and with ſome Alterations for the Better, followed by* Sherwin *in his Mathematical-Tables publiſh'd at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000 *conſiſting of* 7 *Places of Figures. To which are ſubjoined the Differences and proportional Parts, by Means of which may be found eaſily the Logarithms of Numbers to* 10000000, *obſerving at the ſame Time that theſe Logarithms conſiſt only of* 7 *Places of Figures. Here are alſo the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Uſe in practical Mathematicks.*

THE

THE CONSTRUCTION OF Logarithms.

THESE most excellent and useful Numbers were first invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiston* in *Scotland*, aforesaid, (*Ann.* 1614.) who ingeniously contrived to perform Multiplication and Division of Natural Numbers, by only adding or subtracting certain Artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square: by 3 for the Cube: by 4 for the Biquadrat, *&c.*

This Invention of his (no doubt) proceeded from a mature Consideration of the Coherence that is betwixt Numbers in Geometrical Proportion and those in Arithmetical Progression.

As in these following:

$$\textit{Viz.}\left\{\begin{array}{l} 1 \,.\, 2 \,.\, 4 \,.\, 8 \,.\, 16 \,.\, 32 \,.\, 64 \,.\, 128, \textit{ \&c. } \text{Geometrical,} \\ 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7, \textit{ \&c. } \text{Arithmetical.} \end{array}\right.$$

It is very perceptible, that as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Division*, those in the Arithmetical Progression are produced by *Addition* or *Subtraction*: As doth appear in this Example;

$$\textit{Viz.}\left\{\begin{array}{l} 4 \times 32 = 128 \\ 2 + 5 = 7 \end{array}\right\} \text{or} \left\{\begin{array}{l} 128 \div 32 = 4 \text{ Geometr.} \\ 7 - 5 = 2 \text{ Arithmet.} \end{array}\right.$$

$$\text{Again,}\left\{\begin{array}{l} 1 \,.\, 10 \,.\, 100 \,.\, 1000 \,.\, 10000 \,.\, 100000, \textit{ \&c. } \text{Geometr.} \\ 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5, \textit{ \&c. } \text{Arithmet.} \end{array}\right.$$

The same Coherence is betwixt these Latter, as was between the two first Ranks.

$$\textit{Viz.}\left\{\begin{array}{l} 1000 \times 10 = 10000 \\ 3 + 1 = 4 \end{array}\right\} \text{or} \left\{\begin{array}{l} 100000 \div 1000 = 100 \text{ Geometr.} \\ 5 - 3 = 2 \text{ Arithmet.} \end{array}\right.$$

Either of these Examples do sufficiently shew the Reason and very Ground of Logarithms.

And

And from the Latter of these it was, that the Prime Logarithms or Characteristics were first assigned.

As in this Table:

Natural Num.	*Logarithms.*
1	0,0000000
10	1,0000000
100	2,0000000
1000	3,0000000
10000	4,0000000
100000	5,0000000

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers situated betwixt 1 and 10, *viz.* of 2, 3, 4, 5, 6, 7, *&c.* and of those betwixt 10 and 100, *viz.* of 11, 12, 13, 14, 15, *&c.* and so on for the rest. This was a Work of some Difficulty, and very laborious.

The first Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, so as that the last (and least thereof) might be a mixed Number less than 2, and so near 1, as to have such a Number of Cyphers before the significant Figures thereof, as was intended the Places of Logarithms in the Table should consist of. Which Means are to be found, by extracting the Square Root of 10 (having first annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and so by a continued Extraction of Root out of Root, until there be a Root so qualify'd as before-mentioned: Which to make a Table to seven Places in the Logarithms, will require twenty-five several Extractions, the last of which will produce this Number, 1,0000000686238.

The next Step was to find out a Number betwixt (1) and (0) in Arithmetical Progression, that might truly correspond with the Mean before found (betwixt 10 and 1) such a Number must consequently be its Logarithm. And this may be found by a continual bisecting (or halving) of 1, so often as was the Number of the foregoing Extractions, (to wit, twenty-five) the last of which Bisections will produce 0,0000000298023322, *&c.* the true Logarithm of 1,0000000686238.

For as 1,0000000686238 by twenty-five continued Involutions (*viz.* first into it self, then that Product into it self, and so on successively) will produce 10; so will 0,0000000298023 by the like Number of Doublings and Re doublings, produce 1.

This Mean (or Number) and its Logarithm being thus found, it will follow by Proportion,

As

As the Significant Figures of this Mean : *are to the Significant Figures of its Logarithms* :: *So are the Significant Figures of any Mean, betwixt any given Number and* 1 : (having seven Cyphers before such Figures, as this hath) *To the Significant Figures of its Logarithm.* To which must be prefixed seven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its corresponding Mean, will at last discover the true Logarithm of the given Number. For the clearing of this, take an Example.

Suppose it were required to find the Logarithm of the Number 2, to seven Places. First, by a continued Extraction of Root out of Root, beginning at 2, find such a Mean, or Root as before, betwixt 2 and 1, as will have seven Cyphers before its significant Figures; which after twenty-three several Extractions, will be this Number 1,00000008262958. Then according to the foregoing Proportions, it will be

$$6862238 : 2980232 :: 8262958 : 3588557$$

To which prefix seven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, ,00000003588557; which being doubled and redoubled, as abovesaid, will produce 0,30102997958658 the true Logarithm of 2; which being contracted to seven Places, according to the first Design, and agreeable to the seven Places of Cyphers, then it will become 0,3010299. But in all the Tables that I have seen, the Logarithm of 2 is 0,3010300: I conceive the Reason is, because the remaining Figures 7958658 come so near Unity of the last Place in the retained Figures.

And by the same Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, *viz.* of 3. 7. 11. 13. *&c.* (that is, of such Numbers as cannot be produced by the multiplying of two Integer Factors) are obtained, the rest may be easily composed by *Addition* and *Subtraction* only.

For as $3 \times 2 = 6$ So Log. of 3 + Log. of 2. = Log. of 6.
And as $10 \div 2 = 5$ So Log. of 10 — Log. of 2 = Log. of 5

The like of all Numbers that have aliquot Parts, (that is, such Integer Numbers as may be divided by Integers.) And indeed the Logarithms of several of the prime Numbers may also be obtained by *Addition* or *Subtraction*, as might easily be shewed, and is not difficult to conceive by any one, who but duly considers the Nature and Design of Logarithms; *&c.* of which I shall forbear saying any thing in this Place, and keep to my first Design herein, which was to give a brief Account of the ingenious Author's Method as I conceive it of making the same: who undoubtedly found it a very difficult Work, by Reason there is required so many several Extractions of Roots out of Roots, which must needs

needs render it both troublesom and laborious. Then to propose a different Method of raising the Logarithms of such prime Numbers before mentioned, which require the Extraction of Roots to obtain their respective Means, with one tenth Part of the Trouble and Time required by the foregoing Method. And not only so, but more exact; for by our present Method of converging Series, the Root of any Power, how high soever it be, is easily found at one single Extraction; and thereby the Errors which would arise by extracting a *Surd Root* out of a *Surd Root*, especially when often repeated, are avoided; and consequently such a Mean as may be required betwixt any Number and Unity, is thereby more exactly found.

Now how this may be performed, I here intend to shew, as briefly as I can. In order thereunto, take this as a Model.

Let $a =$ the Root, or Mean required betwixt any Number and Unity:

$$\text{Then}\begin{cases} a^2 = \square\, a \;.\; a^4 = \square\, a^2 \;.\; a^8 = \square\, a^4 \;. \\ a^{16} = \square\, a^8 \;.\; a^{32} = \square\, a^{16} \;.\; a^{64} = \square\, a^{32} \\ a^{128} = \square\, a^{64} .\; a^{256} = \square\, a^{128} \;.\; a^{512} = \square\, a^{256} \end{cases}$$

And so on successively with the Indices in Geometrical Progression, until the Power of a be made equal to such a Term in that Progression, as that the Root, or Value of a may have betwixt Unity and its significant Figures, so many Cyphers, as are the intended Number of Places in the Logarithms.

For Instance, Let it be required to find the Mean between 10 and 1, then the Power of a must be $a^{33554432} = 10$, this Index 33554432 being the 25th Term in Geometrical Progression, which may be thus determined.

Let 1, the Characteristic or Logarithm of 10, be divided by such a Term in Geometrical Progression, as will cause such a Number of Cyphers to be before the significant Figures in the Quotient, as are required to be before the Figures of the Root a; suppose 7, as before.

Then $1 \div 33554432 = ,0000000298023$, &c.

which is the true Arithmetical Mean (as before found, by a continual bisecting of 1) correspondent to that signify'd by a And therefore the Value of a found by extracting the respective Root of $10 = a^{33554432}$ will be the Mean required;

Viz. 1,00000006862238 whose Log. is ,0000000298023

These being found, are the Foundation of the rest, as before.

Then suppose it be required to find the Logarithm of any of the Prime Numbers; if you please, that of 2. In order thereunto, let $a =$ the Root or Mean sought betwixt 2 and 1, as before; then must a be continually involved, as by the above Model, until its Index be equal to the greatest Term in Geometrical Progression,

whose

whose Number of Places of Figures are to be equal to the Number of required Cyphers before a, to wit 7. According to which, the Power of a will be $a^{8388608} = 2$ (this 8388608 being the 23d Term in Geometrical Progression) consequently the respective Root of $2 = a^{8388608}$ will be the Mean required.

Example.

Let $r + e = a$

Then will $r^{8388608} + 8388608\, r^{8388607} e$

$+ 35184367894528\, r^{8388606} ee = a^{8388606} = 2$

Suppose $r = 1$

Then $1 + 8388608e + 35184367894528ee = 2$

That is $8388608e + 35184367894528ee = 1$

Each Part being divided by the Co-efficient found prefixed to ee, viz. 351843, &c. then it will become

$,00000023e + ee = ,000000000000000284 = D$

$$Consequently \left\{ \frac{D}{,00000023 + e} \right. = e$$

$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

$,000000000000000284 = D$

$,00000023$	248	($,00000008 = e$
$+ e = ,00000008$		
Divisor $,00000031$	36	

First $r = 1,$

$+ e = ,00000008$

New $r = 1,00000008$

Which being duly involved, in the same Order as the Model denotes, and multiplied into the respective Co-efficients, will then produce these Numbers,

Viz. $1,9563638967 + 16411168e + 68833416066289ee = 2$

Then $16411168e + 68833416066289ee = ,0436361033$

And $,0000002384e + ee = ,00000000000000063393 = D$

$$Consequently \left\{ \frac{D}{,0000002384 + e} \right. = e$$

$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

$,00000000000000063393 = D$

$,0000002384$	480	($,00000000263 = e$
$+ e = ,0000000026$		
Divisor $,000000240$	15393	
	14460	
Divisor $,0000002410$	9330	
	7230	

$$\begin{array}{rl} \text{Last } r = & 1{,}00000008 \\ + e = & {,}0000000263 \\ \hline \text{New } r = & 1{,}0000000826\dot{3} \end{array}$$

I take only 1 0000000268 $= r$; the which being involved, and ordered as before will produce these following Numbers, *viz.*
1,999503684867 + 167730288e + 7035126745408ee = 2
Then 167730288e + 7035126745408ee = ,000496315133
,0000002384186e + ee = 0000000000000000000705481443 = D

$$\textit{Consequently}\left\{\frac{D}{{,}0000002384186 + e} = e\right.$$

,0000000000000000000705481443 = D

	,0000002384186	47686 (,00000000000295 = e
$+ e =$	,00000000000295	
Divisor	,00000023843	2286214
		2146023
Divisor	,000000238447	14019143
		11922405
Divisor	,0000002384481 *	*20967380 (879 = e
		19075848
		1891532
		1669136
		222396
		214596

* Here I desist forming a new Divisor, and make use of the Abridgment.

$$\begin{array}{rl} \text{Last } r = & 1{,}0000000826 \\ + e = & {,}00000000000295879 \\ \hline a = & 1{,}0000000826295879 \end{array}$$

This Value of $a =$ 1,0000000826295879 is the Geometrical Mean betwixt 2 and 1, as was required; (agreeable to that before found, by twenty-three several Extractions.) And by this Method of proceeding, may be found the Mean betwixt 10 and 1, *viz.* 1,0000000686238, or betwixt any other of the (before mentioned) Prime Numbers and Unity, as might easily be shewed. But for Brevity Sake, I shall omit giving more Examples thereof, this one being sufficient (especially to the Ingenious) if well considered, and but once understood, to shew the Nature of, and Manner how, to proceed upon the like Occasion, of finding any proposed Mean. The next Thing will be to find the Logarithm of the Number

Number from whence such Mean was produced, which may be thus performed.

First, find its corresponding Arithmetical Mean, or Logarithm, by Proportion, (as in *Pag.* 462.) Then multiply that correspon-ding Mean, so found, into the Index Number of such Power as the Geometrical Mean was produced from; that Product will be the Logarithm of the given Number (without a continued Dou-bling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having first found 1,0000000686238, the proper Geometrical Mean betwixt 10 and 1; also its corresponding Logarithm ,0000000298023 2 (as before directed) with them and the Mean be-twixt 2 and 1, last found, *viz.* 1,00000000826295879; make use of the above-mentioned Proportion, (as in *Pag.* 463.) *viz.*

6862238 : 2980232 :: 826295879 : 358855729

To which prefix seven Cyphers to complete it (as before.) Then it will become ,0000000358855729. This Number being multiplied into the Power of *a* (what that is, see *Pag.* 465.) will produce the Logarithm of 2.

viz. ,0000000358855729 × 8388608 = 0,30103000391352

But according to the first Design, it is required to have but seven Places, *viz.* 0,301300; which is the true Logarithm of 2 without any Defect.

Thus I have presented you with a new and expeditious Method of making Logarithms; which if they were required to fourteen or fifteen Places (I can modestly say) they might then be made with one twentieth Part of the Time and Trouble required by the first Method.

METHOD III.

A NEW TABLE OF LOGARITHMS.

Composed by Mr. Long.

Finding the Logarithm *by* Division *only, and the Natural Number belonging to a* Logarithm, *by* Multiplication *only.*

Log.	Nat. Num.	Log.	Nat. Num.
0,9	7.943282347	0,00009	1.000207254
0,8	6.309573445	0,00008	1.000184224
0,7	5.011872336	0,00007	1.000161194
0,6	3.981071706	0,00006	1.000138165
0,5	3.162277660	0,00005	1.000115136
0,4	2.511886432	0,00004	1.000092106
0,3	1.995262315	0,00003	1.000069080
0,2	1.584893193	0,00002	1.000046053
0,1	1.258925412	0,00001	1.000023026
0,09	1.230268771	0,000009	1.000020724
0,08	1.202264435	0,000008	1.000018421
0,07	1.174897555	0,000007	1.000016118
0,06	1.148153621	0,000006	1.000013816
0,05	1.122018454	0,000005	1.000011513
0,04	1.096478196	0,000004	1.000009210
0,03	1.071519305	0,000003	1.000006908
0,02	1.047128548	0,000002	1.000004605
0,01	1.023292992	0,000001	1.000002302
0,009	1.020939484	0,0000009	1.000002072
0,008	1.018591388	0,0000008	1.000001842
0,007	1.016248694	0,0000007	1.000001611
0,006	1.013911386	0,0000006	1.000001381
0,005	1.011579454	0,0000005	1.000001151
0,004	1.009252886	0,0000004	1.000000921
0,003	1.006931669	0,0000003	1.000000690
0,002	1.004615794	0,0000002	1.000000460
0,001	1.002305238	0,0000001	1.000000230
0.0009	1.002074475	0,00000009	1.000000207
0.0008	1.001843766	0 00000008	1.000000184
0,0007	1.001613109	0 00000007	1.000000161
0.0006	1.001382506	0,00000006	1.000000138
0,0005	1.001151956	0.00000005	1.000000115
0.0004	1.000921459	0 00000004	1.000000092
0.0003	1.000691015	0,00000003	1.000000069
0.0002	1.000460623	0.00000002	1.000000046
0.0001	1.000230285	0 00000001	1.000000023

This

This Table I sometimes make use of for finding the Logarithm of any Number propos'd, and *vice versa*. Suppose I had Occasion to find the Logarithm of 2000. I look in the first Class of my Table (the whole Table consists of 8 Classes) for the next less to 2, which is 1.995262315, and against it is 3, which consequently is the first Figure of the Logarithm sought. Again, dividing the Number propos'd 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being look'd for in the second Class of the Table, and finding neither its Equal, nor a Lesser, I add 0 to the Part of the Logarithm before found, and look for the said Quotient 1.002374467 in the third Class, where the next less is 1.002305238, and against it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238. last found in the Table, the Quotient is 1.000069070; which being sought in the fourth Class gives 0, but being sought in the fifth Class gives 2, to be added to the Part of the Logarithm already found; and dividing the last Quotient by the Number last found in the Table, *viz.* 1.000046053, the Quotient is 1.000023015, which being sought in the sixth Class, gives 9 to the Part of the Logarithm already found: And dividing the last Quotient by the new Divisor, *viz.* 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115; shews that the Logarithm already found, *viz.* 3.3010299 is less than the Truth by more than half an Unit; wherefore adding 1, you have *Briggs*'s Logarithm of 2000, *viz.* 3.3010300.

If any Logarithm be given, suppose 3,3010300, throw away the Characteristic, then over against these Figures 3...0..1..0..0, you have in their respective Classes 1.995262315.....0..... 1.002305238.....0.....1.000069080....0...0 which multiplied continually into one another, the Product is 2.000000019966, which by reason the Characteristic is 3, becomes 2.000,000019966, *&c.* that is, 2000. the Natural Number desired. I shall not mention the Method by which this Table is fram'd, because you will easily see that from the Use of it.

It is obvious to the intelligent Reader, that these Classes of Numbers are no other than so many Scales of mean Proportionals: In the first Class, between 1 and 10; so that the last Number thereof, *viz.* 1,258925412 is the tenth Root of 10, and the rest in order ascending are the Powers thereof. So in the second Class, the last Number 1.023292992 is the hundredth Root of 10, and the rest in the same Manner are Powers thereof. So 1 002305238 in the third Class, is the tenth Root of the last of the second, and the rest its Powers, *&c.* Or, which is all one, each Number in the preceding Class, is the tenth Power of the corresponding Number in the next following Class: Whence 'tis plain, that to construct

struct these Tables requires no more than one Extraction of the fifth or sursolid Root for each Class, the rest of the Work being done by the common Rules of Arithmetick.

METHOD IV.

Their Construction, according to the common Rules, given by many Extractions of Roots, is tedious; the best Way yet known is this which follows.

To make a Table of Logarithms.

First, Put for the Logarithm of 1, a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms consist of; for 10, an Unit with the same Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, &c.

Secondly, Find the Difference between some two Logarithms above 1000 or rather above 10000, that differ by Unity; thus. Multiply the two Numbers together, and that Product you must multiply again by 43429448190325183896 * which last Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference sought.

Suppose we would find the Difference between the Log. 10000, and 10001, the Product of these two Numbers is 1.00010000, which multiplied by 4343 produceth 43434343; this divided by 10000.5, quotes 4343. Now if to the Logarithm of 10000, which is 4.0000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

Thirdly. Having thus found the Difference of any two Logarithms Differing by Unity, and consequently some of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers Differing by Unity, you shall have the Difference of the Logarithm of those two Numbers.

Thus to find the Difference betwixt the Logarithm of 274, and 275; divide 4343 the Difference of the Logarithm of 10000, and 10001 by 274.5 the Quotient 15821, is the Difference sought.

Fourthly Having by this Means found a few of the prime Logarithms, the rest are made by Addition and Subtraction, and having made

* *Which is the Subtangent of the Curve expressing* Brigg's *Logarithms. See* Keil's *Trig.* Pag. 135. 140, &c.

made the Canon upward, above 1000 to 10000, by Consequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms must be found in first Place, are these, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, *&c.* or the same Numbers with Cyphers.

But since it was very tedious and laborious, to find the Logarithms of the Prime-Numbers, and not easy to compute Logarithms by Interpolation, by first, second and third, *&c.* Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator* *Gregory* *Wallis* and lastly, Dr. *Halley*, have published infinite converging Series by which the Logarithms of Numbers to any Number of Places, may be had more expeditiously and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found easily and expeditiously, the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Number $z-1$ and $z+1$ be even, and accordingly their Logarithms and the Difference of the Logarithms will be had, which let be called y: Therefore, also the Logarithm of a Number which is a Geometrical-Mean between $z-1$ and $z+1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$ *&c.* shall be equal to the Logarithm of the Ratio, which the Geometrical-Mean between the Numbers $z-1$ and $z+1$, has to the Arithmetical-Mean, *viz.* to the Number z.

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2. with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0,0000434272 7687. And if this Difference be divided by $4z$, or 80004, the Quo-

Quotient $\frac{y}{4z}$ ſhall be ———————————— 0.00000 0000542813

And if the Logarithm of the Geometrical-Mean be added to the Quotient, the Sum will be the Logarithm of .20001. Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Tables, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000, are to be found by this Way, the Labour of doing them, will moſtly conſiſt in ſetting down the Numbers. *Note*. This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a Mind to be inform'd more in this Matter, let them conſult his abovenam'd Treatiſe.

	4.30105 1709302416
	4.30105 1709845230

Mr. *WARD's* *Eaſy* Method of making the Canon of Sines, Tangents, &c.

FIRST, Let me premiſe two Things, that the Periphery of a Circle, whoſe Radius is Unity or 1, is 6.283185, *&c.* and that the Natural Sine of one Minute doth *ſo inſenſibly* differ from the Length of the Arch of one Minute, *that* it may be taken for the ſame.

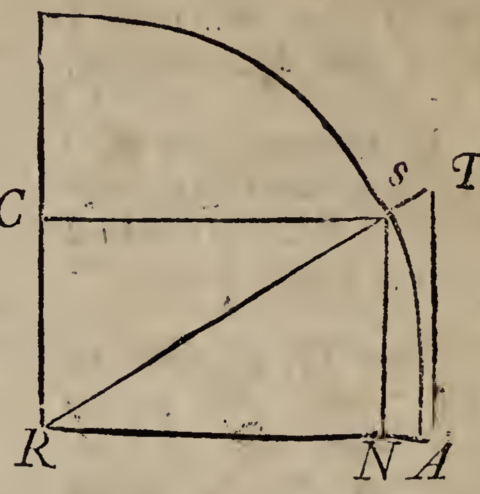
Conſequently, { *As the Periphery in Minutes : Is to the Periphery in equal Parts of the Radius :: So is one Minute : To the Parts agreeing to that Minute.* }

That is, 21600′ : 6,283185 :: 1′ : 0.000290888 = the Natural Sine of one Minute; which agrees with the largeſt Table of Sines I ever ſaw.

Having thus got the Sine of one Minute, its Co-Sine may be thus found:

Suppoſe

Suppose $RA = RS$ the Radius of any Circle, $SN =$ the Sine of the Arch SA. Then $RN = CS$ is the Co-Sine of that Arch. But $\square RS - \square SN = \square RN$. consequently $\sqrt{\square RS - \square SN} = RN$.



That is, From the Square of the Radius, subtract the Square of the Sine of 1′, the Square Root of the Remainder will be the Co Sine of 1′, per *Chap. 9. Prop. 1.* In Numbers, the Sine of 1′ is 00029088, its Square is 0,00000008461 2; and 1 — 0,00000008461 2 = 0,99999991538 8, the Square Root thereof is ,99999995 = the Co-Sine required.

The Sine and Co-Sine of one Minute being thus obtain'd, all the rest of the Sines in the Quadrant may be gradually calculated by Mr. *Michael Dary*'s Sinical Proportions; which I shall here insert, to the same Effect as they are in his Miscellanies; and then explain and demonstrate the Truth of those Proportions.

If a Rank of Arches be equi-different;

Then { *As the Sine of any Arch in that Rank : Is to the Sum of the Sines of any two Arches equally remote from it on each Side : : So is the Sine of any other Arch in the said Rank : To the Sum of the Sines of two Arches next it on each Side; having the same common Distance.*

Immediately after these Proportions, he lays down the following Equations:

Three Arches equi-different, being proposed; if (saith he) you put $Z =$ the Sine of the greater Extreme, $y =$ the Sine of the lesser Extreme; $M =$ the Sine of the Mean; $m =$ the Co-Sine thereof; D the Sine of the common Difference; $d =$ the Co-sine thereof; and $R =$ the Radius.

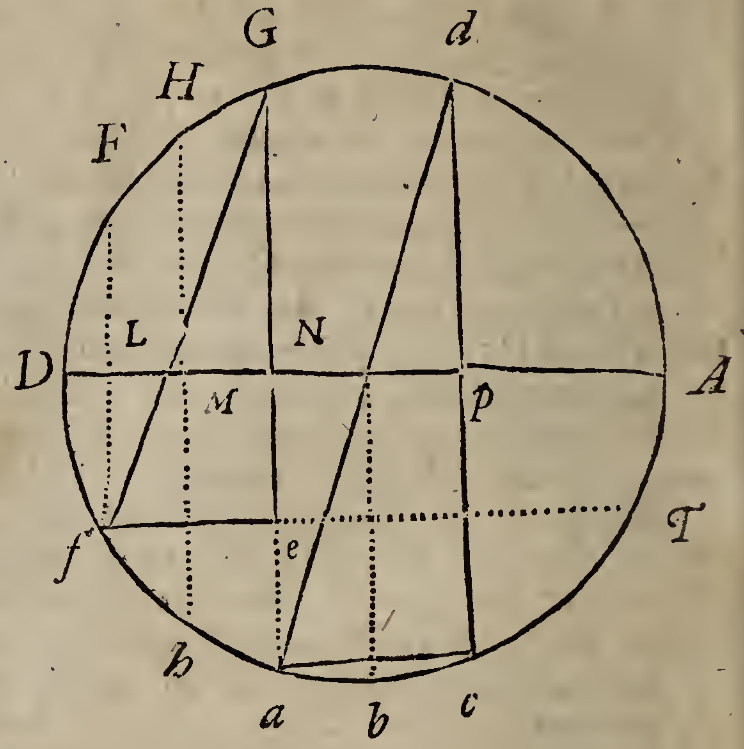
1\. Then $Z + y = \frac{2Md}{R}$. 2. Then $Z - y = \frac{2mD}{R}$.

3\. Then $Zy = MM - DD$. 4. Then $\frac{Z}{y} = \frac{Md + mD}{Md - Md}$.

From the foregoing, it is evident, (saith he) that if two Thirds, *viz.* either the former or latter 60 Degrees, or the former 30 Degr. and the latter 30 Degr. of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

 Thus

Thus far is from the Ingenious Mr. *Dary*, concerning these excellent Proportions; the Truth whereof I shall thus demonstrate.



In the annexed Circle $DA = da$ are Diameters, $fh = ha = ab = bc$, are equal Arches.

Draw fT parallel to DA; then will $Ne = Lf$. And the $\triangle dac$, like the $\triangle Gfe$, being both right-angled at c and e, and $\angle d = \angle G$ because subtended by the equal Arches $ac = fa$.

Therefore $da : dc :: Gf : Ge$.

Consequently $\frac{1}{2}da : dc :: \frac{1}{2}Gf : Ge$. But $Hb = Gf$, whence $HM = \frac{1}{2}Gf$, and $\frac{1}{2}da =$ the Radius, $dp = \frac{1}{2}dc$. Therefore it will be, Radius : $2dp :: HM = \frac{1}{2}Gf : GN + Ne = GN + Lf$. That is, As the Radius : Is to twice the Sine dp :: So is the Sine HM : To the Sum of the two Sines GN and $FL = fL$. Q. E. D.

I shall now explain these Proportions, and shew how they may be applied in Practice: Having the Sine of one Minute, and its Co-Sine as before; let the Radius be made the Mean or Middle Term between those two Extremes; then the Proportions will run

Thus, { *As the Radius : Is to the double Co-sine of one Minute :: So is the Sine of one Minute : To the Sine of two Minutes, and of* 00′ : *And so is the Sine of* 2′ :: *To the Sum of the Sines of* 3′ *and* 1′ :: *And so is Sine of* 3′ : *To the Sum of Sines of* 4′ *and* 2′.

And so on in a successive Order, from Minute to Minute.

And then, if from the Sum of the Sines of 3′ and 1′ be taken the Sine of 1′, the Remainder will be the Sine of 3′: And the like, if from the Sum of the Sines of 4′ and 2′, be taken the Sine of 2′, there will remain the Sine of 4′, &c.

Proceeding

Proceeding on by this Method, all the Natural Sines in the Quadrant may be easily calculated by Addition and Subtraction only. For the Radius, or first Term in the Proportion, being 1.0000000 or Unity, Division is wholly avoided: And because the second Term in the Proportion varies not, if a Tariffa, or small Table be made thereof, to all the nine Digits, then Multiplication is also avoided. For by the help of that Tariffa, the whole Work may be perform'd by Additon and Subtraction, until all the Sines are gradually made.

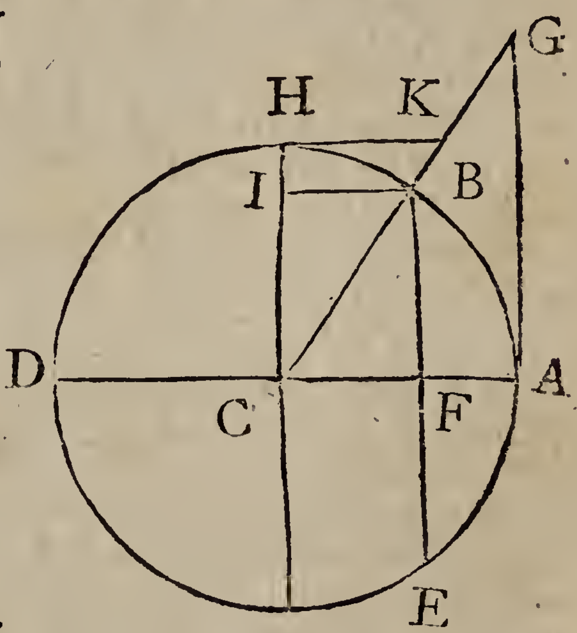
Thus you have an easy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following

Proportions $\left\{\right.$ *As the Co-sine of any Arch : Is to the Sine of that Arch : : So is the Radius : To the Tangent of the same Arch.*

That is, by *the first Scheme of this Problem*, $RN : SN :: RA : TA$. And $RN : RS :: RA : RT =$ the Secant of that Arch.

Plane Trigonometry.

Definitions.



1 A Circle is suppos'd to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds, &c. Any Portion of whose Circumference is called an Arch, and is measured by the Number of Degrees it contains.

2. A Chord or Subtense is a strait Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

3. A Sine (or Right sine) is a strait Line drawn from one End of an Arch perpendicular to that Diameter passing thro' the other End; or it is half the Chord of twice the Arch; so BF is the Sine of the Arches BA, BD. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greatest of all Sines, the Sine of an Arch greater than a Quadrant being less than the Radius.

4. The

4. The Difference of an Arch from a Quadrant, whether it be greater or leſs, is called its Complement; ſo HB is the Complement of the Arches BA, BD; BI is the Sine of that Complement, and therefore it is called the Co-ſine, or Sine-Complement of the Arches BA. BD.

5. The Secant of an Arch is a ſtrait Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which is a ſtrait Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arch; ſo CG is the Secant, and AG the Tangent of the Arches BA. BD: And CK is the Co-ſecant, and HK the Co-tangent of the ſaid Arches.

6. A Verſed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus FA is the Verſed Sine of the Arch BA; and FD of the Arch BD.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-ſine; thus CF is = IB.

9. If an Arch be greater or leſs than a Quadrant, the Sum or Difference of the Radius and Co-ſine is equal to the Verſed Sine.

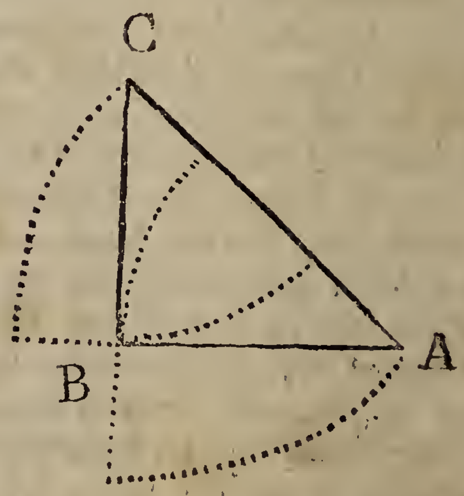
In a Triangle are ſix Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a Plane Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, ſuppoſing the Radius divided into any Number of equal Parts, we know how many of thoſe equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd: The Art of inferring which is called *Trigonometry*, and it is either Plane or Spherical.

Plane Trigonometry is ſolv'd by the Help of four fundamental Propoſitions, call'd *Axioms*.

Axiom I.

In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then ſhall the other Leg CB or AB be the Tangent of the Angle oppoſite to it, and the Hypothenuſe AC (or Side oppoſite to the Right-angle) its Secant (by Definition 5.)

But if the Hypothenuſe AC be made the Radius of a Circle, then will the Legs (or Sides including the Right angle) to wit CB and AB be the Sines of the Angles oppoſite (by Definition 3.)



Upon

Upon this *Axiom* depends the Solution of the ſeven Caſes of Right-angled Plane Triangles.

Note, That the three Angles of a Plane Triangle make two Right Angles. or 180 Degrees, by 32. 1 *Eucl.*

For the more eaſy making the Proportions for the Solution of Right-angled Triangles, obſerve, that as different Sides are made Radius, ſo the other Sides acquire different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then ſay, As the Name of the Side given, is to the Name of the Side required; ſo is the Side given, to the Side required.

But to find an Angle, one of the given Sides muſt be made Radius; then, As the Side made Radius, is to the other Side; ſo is the Name of the firſt Side (which is Radius) to the Name of the ſecond Side; which fourth Proportional muſt be found among the Sines or Tangents, &c. to be determin'd by the Side made Radius, and againſt it is the Angle required.

The Proportions for the Solution of ſeven Caſes of Plane Right-angled Triangles. [*See the next foregoing* Fig.]

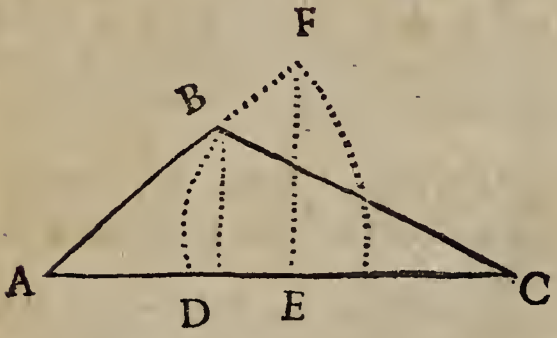
Given	*Req'd.*	*Proportions.*	*Rad.*	*Caſe.*
A B A and C	B C	Coſi. A : Si. A :: AB : BC. R : Tan. A :: AB : BC. Co-t. A : R :: AB : BC.	AC AB BC	1
A B A and C	A C	Coſi. A : R :: AB . AC. R : Sec. A :: AB : AC. Tan. A : Coſe. A :: AB : AC.	AC AB BC	2
A B B C	A and C	AB : BC :: R : Tan. A. Complement is C. BC : AB :: R : Tan. C. Complement is A.	AB BC	3
A B B C	A C	AB : BC :: R : Tan. A; Then Coſi. A : R :: AB : AC. or √ : □ AB + □ BC : = AC (*per* 47. 1. *Eucl.*	AB AC	4
A B A C	A and C	AC : BC :: R : Coſi. A. AB : AC :: R : Secant A.	AC AB	5
A B A C	B C	AC : AB :: R : Coſi. A; Then R : Tan. A :: AB : BC, or √ : □ AC — □ AB : = BC.	AC AB	6
A C A and C	A B	R : Coſi. A :: AC : AB. Sec. A : R :: AC : AB. Coſ. A : Cot. A :: AC : AB.	AC AB BC	7

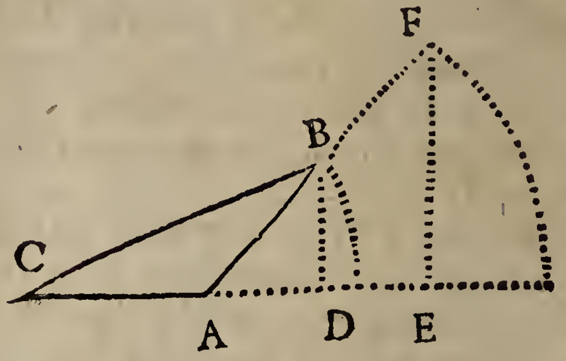
Axiom

Axiom II.

In any Triangle the Sides are proportional to the Sines of the opposite Angles.

Demonstration.





Produce the lesser Side of the Triangle ABC, to wit AB to F, making AF = BC: Let fall the Perpendiculars BD, FE, upon the Side CA produc'd, if need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius BC = AF.

Now the Triangles ABD and AFE, having the ∠ A common to them both, and the ∠ D = ∠ E = to a Right-angle, are similar; wherefore (by 4. 6 *Eucl. Elem.*) AF (BC) : AB :: FE : BD; *viz.* :: Si. A : Si. C. *Q. E. D.* Otherwise thus;

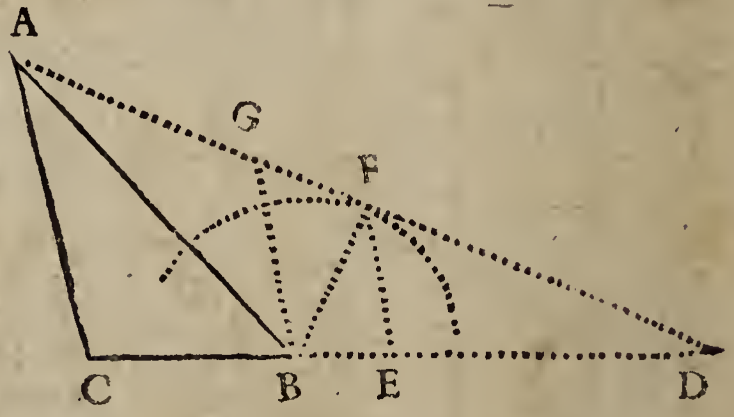
By *Ax.* I. AB : R :: BD : Si. A, and BC : R :: BD : Si. C;
Therefore AB × Si. A (= R × BD) = BC × Si. C;
Wherefore AB : BC :: Si. C : Si. A. *Q. E. D.*

Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle, is to their Difference, as the Tangent of half the Sum of the Angles opposite to those Legs, is to the Tangent of half their Difference.

Demonstration.

In the Triangle ABC produce CB, the lesser Leg of the Angle B, till BD becomes = BA, the greater Leg, and then bisect CD in E; join AD and bisect it also in F; draw BF, which (by 8. 1 *Eucl. El.*) will be perpen. to A D; and draw EF, which (by 2. 6 *Eucl. Elem.*) will be parallel to AC. Then will the Angle ABF = FBD = ½ABD, which external Angle ABD is (by 32. 1 *Eucl. Elem.*) = BAC + C, that is to Sum of the opposite Angles required.



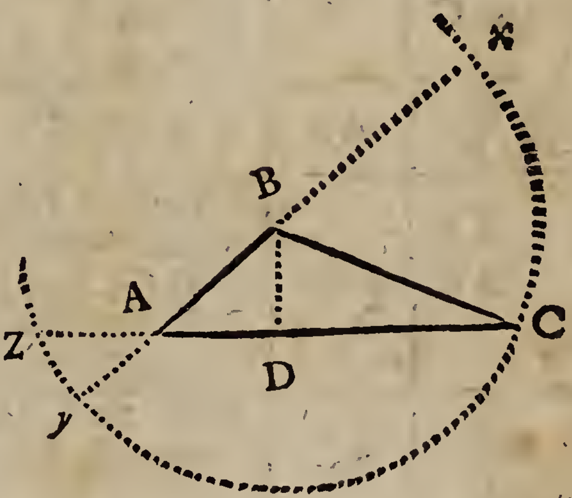
Draw then BG parallel to CA, so will the Angle GBA be (by 29. 1 *Eucl. Elem.*) equal to its Alternate one BAC; and if from half the

Sum

Sum of the oppoſite Angles you take the leſſer Angle, *i. e.* If from $\angle$ ABF you take the $\angle$ GBA, there will remain $\angle$ GBF = half the Difference of the oppoſite Angles: And ſo alſo, if from CE half the Sum of the Legs, you take CB the leſſer Leg, there will remain BE equal to half the Difference of the Legs. And then, ſince the $\triangle$ ABF is Right-angled, if BF be made Radius, AF will be the Tangent of $\angle$ ABF (*i. e.* the Tangent of half the Sum of the oppoſite Angles); and in the little $\triangle$ GBF, FG will be the Tangent of the $\angle$ GBF (*i. e.* the Tangent of half the Difference of the oppoſite Angles): But the Segments of the Legs of any Triangle cut by Lines parallel to the Baſe, being (by *Schol.* to 2. 6 *Eucl. El.*) proportional; EC : EB :: FA : FG; that is in Words, half the Sum of the Legs, is to half their Difference, as the Tangent of half the Sum of the oppoſite Angles, is to the Tangent of half their Difference: But Wholes are as their Halves; wherefore the Sum of the Legs, is to their Difference, as the Tangent of half the Sum of the Angles oppoſite, is to the Tangent of half their Difference. *Q. E. D.*

Axiom IV.

The Baſe, or greateſt Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs, is to the Difference of the Segments of the Baſe made by a Perpendicular let fall from the Angle oppoſite to the Baſe.



Demonſtration.

From the $\angle$ B, on the Baſe AC, of the $\triangle$ ABC, let fall the Perpendicular BD; on B as a Center, with the greater Leg BC, as a Radius, deſcribe the Circle B*x*C*y*Z; and produce AB to *x* and *y*, and CA to Z. Then,

By the 35. 3 *Eucl. Elem.* A*y* × A*x* is = AC × AZ; viz. : BC — BA : × : BC + BA : = AC × : DC — DA :

Therefore AC : BC + BA : : BC : — BA : DC — DA. *Q. E. D.*

Otherwiſe,

Let the Difference of the Squares of the Sides BC and AB be taken and divided by the Baſe AC, the Quotient ſhall be the Difference of the Segments of the Baſe aforeſaid:

Or,

Square all the 3 Sides, and deduct the Square of one of the leſs Sides out of the Sum of the other two Squares, divide half the Remainder by the longeſt Side, the Quotient is the Alternate Segment of the Baſe.

The Proportions for the Solution of the ſix Caſes of Plane oblique Triangles. [*See the laſt* Fig.]

Given

Given.	Reqd.	Proportions.	Ax.	Case.
A B B C and C	A	AB : BC :: Si. C : Si. A.	2	1

N. B. 1st, If the given Angle be Obtuse, the other 2 Angles then are each Acute.

2dly, If the Side opposite to the given Angle is longer than the Side opposite to the Angle sought, then is the Angle sought Acute, but if shorter, then is the said Angle doubtful, and may be either Acute or Obtuse, because both the Sine and its Complement to two Right Angles are the same: Wherefore to be certain, of what Quality the Angle opposite to the greatest Side is. Take the Sum and Difference of the greatest Side and Middle (or least) and their Logarithms; if the half of them be equal to the Logarithm of the third Side, the Angle opposite to the greatest Side is a Right Angle, but if the Logarithm of the third Side be greater than the half it is Acute, if less, it is Obtuse: Or, without Logarithms, multiply the said Sum by the Difference abovesaid, and extract the Square-Root,

which if { Equal to / Greater than / Less than } the third Side, then is the greatest Angle { Right / Obtuse / Acute }

Given.	Reqd.	Proportions.	Ax.	Case.
A B B C and C	A C	AB : BC :: Si. C : Si. A. Hence, by Subtraction, the ∠ B will be known. Si. A : Si. B :: BC : AC.	2	2
A, C and BC	AB	Si. A : Si. C :: BC : AB.	2	3
B A B B C	A and C	BC + AB : BC − AB :: Tan. $\frac{1}{2}$ Sum of the ∠s opposite : Tan. $\frac{1}{2}$ Difference of the ∠s opposite. Then $\frac{1}{2}$ Sum + $\frac{1}{2}$ Difference = greater ∠ A; and $\frac{1}{2}$ Sum − $\frac{1}{2}$ Difference = lesser ∠ C.	3	4
B AB BC	AC	First, find the Angles by the last; then Si. C : Si. B :: AB : AC.	3 2	5
AB BC AC	A B C	AC : BC + BA :: BC − BA : DC − DA : Then $\frac{1}{2}$ AC + $\frac{1}{2}$ DC − $\frac{1}{2}$ DA = DC. And $\frac{1}{2}$ AC − : $\frac{1}{2}$ DC − $\frac{1}{2}$ DA : = DA. Then AB : AD :: R : Cosi. A. And, CB : DC :: R : Cosi. C. And 180° − ∠ A − ∠ C = ∠ B.	4 1 1	6

Or more readily at one Operation.

From half the Sum of the Sides subduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side subtending the enquired Angle be subducted from the Sum of the Log. of the other Difference and the doubled Radius, half the Remainder shall be the Log. of the Tangent of half the enquired Angle.

Agreeable to this Axiom *in* Gellibrand's Trig. Britannica. p. 46.

As the Rectangle of half the Sum of the Sides and the Difference between that half Sum and the Side opposite to the Angle required, is to the Rectangle of the other two Remainders; so is the Square of Radius, to the Square of the Tangent of half the Angle sought.

Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphæricis assequi, Summa Gloria Mathematici est: Sic enim Cœlum & Terras & Maria felici & admirando calculo Mensurat. Fran. Vieta.

F I N I S.